V. *On the Locus of Singular Points and Lines which occur in connection with the Theory of the Locus of Ultimate Intersections of a System of Surfaces.*

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INTRODUCTION.

In a paper "On the c- and p-Discriminants of Ordinary Integrable Differential Equations of the First Order," published in vol. 19 of the 'Proceedings of the London Mathematical Society,' the factors which occur in the c-discriminant of an equation of the form $f(x, y, c) = 0$, where $f(x, y, c)$ is a rational integral function of x, y, c, are determined analytically.

It is shown* that if $E = 0$ be the equation of the envelope locus of the curves $f(x, y, c) = 0$; if $N = 0$ be the equation of their node-locus; if $C = 0$ be the equation of their cusp-locus, then the factors of the discriminant are E, N^2, C^3.

The singularities considered are those whose forms depend on the terms of the second degree only, when the origin of coordinates is at the singular point.

The object of this paper is to extend these results to surfaces.

It is well known that if the equation of a system of surfaces contain arbitrary parameters, and if a locus of ultimate intersections exist, then there cannot be more than two independent parameters.

Hence the investigation falls naturally into two parts: the first is the case where there is only one independent parameter, and the second is the case where there are two.

The investigation given in this paper is limited to the case in which the equation is rational and integral, both as regards the coordinates and the parameters.†

* The theorem was originally given by Professor Cayley in the 'Messenger of Mathematics,' vol. 2, 1872, pp. 6–12.

† An abstract of the contents of this paper has been printed in the Proceedings, vol. 50, pp. 180–186. A table of contents will be found below, pp. 274–278.

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PART I.—THE EQUATION OF THE SYSTEM OF SURFACES IS A RATIONAL INTEGRAL FUNCTION OF THE COORDINATES AND ONE ARBITRARY PARAMETER.

SECTION I. (Arts. 1–6).—THE FACTORS OF THE DISCRIMINANT WHICH IN GENERAL CORRESPOND TO ENVELOPE AND SINGULAR LINE LOCI.

Art. I.—*To show that if $E = 0$ be the equation of the Envelope Locus, the Discriminant contains E as a factor.*

Let the equation be

$$f(x, y, z, a) = 0 \quad \ldots \quad (1),$$

where x, y, z are the coordinates, a the parameter, and f is supposed to be a rational integral function of x, y, z, a.

Denoting partial differentiation when x, y, z, a are treated as independent variables by D, the locus of ultimate intersections can be obtained by eliminating a between (1) and

$$\frac{\mathrm{D}f(x, y, z, a)}{\mathrm{D}a} = 0 \quad \ldots \quad (2).$$

Let the roots of (2) treated as an equation in a be $a_1, a_2, \ldots$, which will at first be supposed to be all different, so that they do not make

$$\frac{\mathrm{D}^2 f(x, y, z, a)}{\mathrm{D}a^2} = 0.$$

Then if R be a factor introduced to make the discriminant Δ, obtained by eliminating a between (1) and (2) of the proper order and weight,

$$\Delta = \mathrm{R}f(x, y, z, a_1)f(x, y, z, a_2) \ldots \quad (3).$$

Let $x = \xi$, $y = \eta$, $z = \zeta$ satisfy (1) and (2) when $a = \alpha$.

Suppose that a_1 becomes α, when $x = \xi$, $y = \eta$, $z = \zeta$, therefore,

$$f(\xi, \eta, \zeta, \alpha) = 0 \quad \ldots \quad (4),$$

$$\frac{\mathrm{D}f(\xi, \eta, \zeta, \alpha)}{\mathrm{D}\alpha} = 0 \quad \ldots \quad (5).$$

Now in Δ put $x = \xi$, $y = \eta$, $z = \zeta$; and consequently $a_1 = \alpha$, therefore $f(x, y, z, a_1)$ becomes $f(\xi, \eta, \zeta, \alpha)$ and consequently vanishes.

Therefore Δ vanishes when $x = \xi$, $y = \eta$, $z = \zeta$.

The next step is to show that the locus of ultimate intersections is the envelope.

Write, for brevity,

$$\Delta = \mathrm{Q}f(z, y, z, a_1) = \mathrm{Q}f_1 \quad \ldots \quad (6).$$

Now denoting differentiation when x, y, z are the only independent variables by ∂,

$$\frac{\partial \Delta}{\partial x} = \frac{\partial Q}{\partial x} f_1 + Q \left(\frac{Df_1}{Dx} + \frac{Df_1}{D\alpha_1} \frac{\partial \alpha_1}{\partial x} \right).$$

Hence, since $\alpha = \alpha_1$ satisfies (2)

$$\frac{\partial \Delta}{\partial x} = \frac{\partial Q}{\partial x} f_1 + Q \frac{Df_1}{Dx} \quad \ldots \ldots \ldots \quad (7),$$

assuming that $Df_1/D\alpha_1 \; \partial\alpha_1/\partial x$ vanishes when $Df_1/D\alpha_1$ vanishes.

Now when $x = \xi$, $y = \eta$, $z = \zeta$, $f_1 = 0$, therefore

$$\frac{\partial \Delta}{\partial x} = \left[Q \frac{Df_1}{Dx} \right]_{\substack{x=\xi \\ y=\eta \\ z=\zeta}} \quad \ldots \ldots \ldots \quad (8).$$

Hence when $x = \xi$, $y = \eta$, $z = \zeta$

$$\frac{\partial \Delta}{\partial x} \Big/ \frac{Df_1}{Dx} = \frac{\partial \Delta}{\partial y} \Big/ \frac{Df_1}{Dy} = \frac{\partial \Delta}{\partial z} \Big/ \frac{Df_1}{Dz} \quad \ldots \ldots \ldots \quad (9).$$

Now the tangent plane to the surface

$$f(x, y, z, \alpha) = 0 \quad \ldots \ldots \ldots \quad (10)$$

at the point $x = \xi$, $y = \eta$, $z = \zeta$ is

$$(X - \xi) \frac{Df}{D\xi} + (Y - \eta) \frac{Df}{D\eta} + (Z - \zeta) \frac{Df}{D\zeta} = 0 \quad \ldots \ldots \quad (11).$$

Now $Df/D\xi$ stands for $Df(x, y, z, \alpha)/Dx$, when $x = \xi$, $y = \eta$, $z = \zeta$. And the value of Df_1/Dx, *i.e.*, $Df(x, y, z, \alpha_1)/Dx$, when $x = \xi$, $y = \eta$, $z = \zeta$, and, therefore, $\alpha_1 = \alpha$ is the same as the value of $Df(x, y, z, \alpha)/Dx$, when $x = \xi$, $y = \eta$, $z = \zeta$.

This may be expressed thus:—

$$\frac{\partial \Delta}{\partial \xi} \Big/ \frac{Df}{D\xi} = \frac{\partial \Delta}{\partial \eta} \Big/ \frac{Df}{D\eta} = \frac{\partial \Delta}{\partial \zeta} \Big/ \frac{Df}{D\zeta}.$$

Hence the tangent planes to the surfaces $\Delta = 0$, $f(x, y, z, \alpha) = 0$ at the point ξ, η, ζ coincide.

This proves the envelope property in general for the locus of ultimate intersections.

Hence Δ vanishes, if in it x, y, z be made respectively equal to ξ, η, ζ, the coordinates of any point on the envelope-locus.

Therefore Δ contains E as a factor.

But the conclusion fails if

$$\frac{\mathrm{D}f}{\mathrm{D}\xi} = 0, \quad \frac{\mathrm{D}f}{\mathrm{D}\eta} = 0, \quad \frac{\mathrm{D}f}{\mathrm{D}\zeta} = 0 \quad \ldots \ldots \ldots \quad (12).$$

Hence the work itself suggests the examination of this exceptional case, *i.e.*, where a locus of singular points or lines exists.

Example 1.—*Envelope Locus.*

Let the surfaces be

$$[\phi(x, y, z) - a]^2 + \chi(x, y, z) = 0.$$

(A.) *The Discriminant.*

The discriminant is found by eliminating a between the above equation, and

$$-2[\phi(x, y, z) - a] = 0.$$

Hence the discriminant is $\chi(x, y, z)$.

Hence the locus of ultimate intersections is

$$\chi(x, y, z) = 0.$$

(B.) *The envelope locus is* $\chi(x, y, z) = 0$.

For let ξ, η, ζ be any point on $\chi(x, y, z) = 0$.

Let $a = \phi(\xi, \eta, \zeta)$, and consider the single surface

$$[\phi(x, y, z) - \phi(\xi, \eta, \zeta)]^2 + \chi(x, y, z) = 0.$$

Put $x = \xi + \mathrm{X}$, $y = \eta + \mathrm{Y}$, $z = \zeta + \mathrm{Z}$.

Then the lowest terms in X, Y, Z are

$$\mathrm{X}\frac{\partial\chi}{\partial\xi} + \mathrm{Y}\frac{\partial\chi}{\partial\eta} + \mathrm{Z}\frac{\partial\chi}{\partial\zeta}.$$

Hence the surface considered touches $\chi(x, y, z) = 0$ at ξ, η, ζ.

Hence $\chi(x, y, z) = 0$ is the envelope.

It touches the surface at every point of the curve

$$\chi(x, y, z) = 0,$$
$$\phi(x, y, z) = \phi(\xi, \eta, \zeta).$$

Hence this curve is the characteristic.

Art. 2.—*To prove that the Locus of Conic Nodes of the Surfaces $f(x, y, z, a) = 0$ is a Curve, not a Surface.*

At every point of the locus of conic nodes the equations

$$f(x, y, z, a) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

$$\frac{\mathrm{D}f(x, y, z, a)}{\mathrm{D}x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

$$\frac{\mathrm{D}f(x, y, x, a)}{\mathrm{D}y} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

$$\frac{\mathrm{D}f(x, y, z, a)}{\mathrm{D}z} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

are simultaneously satisfied.

In general these are satisfied by a finite number of values of x, y, z, a only. Hence there are only a finite number of conic nodes.

The next case is that in which equations (13)–(16) are equivalent to three independent equations only, and then it is possible to satisfy them by relations of the form

$$x = \phi(a), \qquad y = \psi(a), \qquad z = \chi(a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

In this case there is a curve locus of conic nodes. But as such a locus is defined by two equations, it cannot be determined by equating a factor of the discriminant to zero.

The next case is that in which equations (13)–(16) are equivalent to two independent equations only. Eliminating a between these, the equation of a surface is obtained. This is the case which will be further examined, and it will be shown that the tangent cone at every conic node must break up into two planes, *i.e.*, the conic node becomes a binode.*

Let ξ, η, ζ be the conic node on the surface (10).

Let $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$ be the conic node on the consecutive surface

$$f(x, y, z, \alpha + \delta\alpha) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

Then

$$f(\xi, \eta, \zeta, \alpha) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

$$\frac{\mathrm{D}f(\xi, \eta, \zeta, \alpha)}{\mathrm{D}\xi} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

* In this connection may be noticed Art. 11, in which it is proved that if a surface have upon it a line at every point of which there is a conic node, then the tangent cone at every conic node must break up into two planes so that the line is a binodal line.

$$\frac{Df(\xi, \eta, \zeta, \alpha)}{D\eta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

$$\frac{Df(\xi, \eta, \zeta, \alpha)}{D\zeta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22),$$

and the equations obtained from (19)–(22) by changing ξ, η, ζ, α into $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$, $\alpha + \delta\alpha$ respectively.

Denoting differential coefficients by brackets containing the independent variables, with regard to which the differentiations are performed, these last equations become by means of (19)–(22)

$$[\alpha] (\delta\alpha) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23),$$

$$[\xi, \xi] (\delta\xi) + [\xi, \eta] (\delta\eta) + [\xi, \zeta] (\delta\zeta) + [\xi, \alpha] (\delta\alpha) = 0 \quad . \quad . \quad . \quad (24),$$

$$[\eta, \xi] (\delta\xi) + [\eta, \eta] (\delta\eta) + [\eta, \zeta] (\delta\zeta) + [\eta, \alpha] (\delta\alpha) = 0 \quad . \quad . \quad . \quad (25),$$

$$[\zeta, \xi] (\delta\xi) + [\zeta, \eta] (\delta\eta) + [\zeta, \zeta] (\delta\zeta) + [\zeta, \alpha] (\delta\alpha) = 0 \quad . \quad . \quad . \quad (26).$$

By (23),

$$[\alpha] = 0$$

at every point of the conic node locus. Hence the co-ordinates of every point on the conic node locus satisfy the equation of the locus of ultimate intersections.*

Further, since $[\alpha] = 0$ at every point on the conic node locus, the corresponding equation is satisfied at the conic node on the surface (18).

Hence

$$[\alpha, \xi] (\delta\xi) + [\alpha, \eta] (\delta\eta) + [\alpha, \zeta] (\delta\zeta) + [\alpha, \alpha] (\delta\alpha) = 0 \quad . \quad . \quad . \quad (27).$$

Since equations (24)–(27) must give consistent values for $\delta\xi : \delta\eta : \delta\zeta : \delta\alpha$, it follows that the Jacobian

$$\frac{D\{[\xi], [\eta], [\zeta], [\alpha]\}}{D\{\xi, \eta, \zeta, \alpha\}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

* I am indebted to Dr. FORSYTH for the following example:—

Let the surfaces be

$$(x - a)^2 + (y - a)^2 - \kappa^2(z - a)^2 = 0.$$

The discriminant is

$$\kappa^2(2z - x - y)^2 + (\kappa^2 - 2)(x - y)^2.$$

Hence the locus of ultimate intersections consists of the two planes

$$\kappa(2z - x - y) = \pm \sqrt{2 - \kappa^2}(x - y).$$

These planes intersect in the straight line $x = y = z$, which is the locus of conical points of the surfaces.

If this were the only relation between these equations, they would determine the ratios $\delta\xi : \delta\eta : \delta\zeta : \delta\alpha$.

Hence there would be a curve locus, not a surface locus of conic nodes.

If, then, there be a surface locus, equations (24)–(27) must be equivalent to two independent equations only.

Expressing that (24)–(26) are equivalent to two independent equations only, it follows that

$$\frac{D\{[\xi], [\eta], [\zeta]\}}{D\{\xi, \eta, \zeta\}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

But this is the condition that the tangent cone at the conic node should break up into two planes, and then the conic node becomes a binode.

Hence there cannot be a surface locus of conic nodes, unless the conic nodes become binodes.

Since equations (13)–(16) are equivalent to two independent equations only, every point on the intersection of the surfaces represented by (13) and (14) is a binode on the surface (13).

Hence the surface (13) has a binodal line.

The locus of these binodal lines is a surface at every point of which equations (1) and (2) are satisfied, hence it is a part of the locus of ultimate intersections, and its equation can be determined by equating a factor of the discriminant to zero.

Art. 3.—*To find the conditions which hold at every point on a Surface Locus of Binodal Lines.*

In this case (29) holds.

Hence, in order that (24)–(26) may give finite values for $\delta\xi : \delta\eta : \delta\zeta : \delta\alpha$,

$$\frac{D\{[\xi], [\eta], [\zeta]\}}{D\{\xi, \eta, \alpha\}} = 0 = \frac{D\{[\xi], [\eta], [\alpha]\}}{D\{\xi, \eta, \zeta\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30),$$

$$\frac{D\{[\xi], [\eta], [\zeta]\}}{D\{\xi, \zeta, \alpha\}} = 0 = \frac{D\{[\xi], [\zeta], [\alpha]\}}{D\{\xi, \eta, \zeta\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31),$$

$$\frac{D\{[\xi], [\eta], [\zeta]\}}{D\{\eta, \zeta, \alpha\}} = 0 = \frac{D\{[\eta], [\zeta], [\alpha]\}}{D\{\xi, \eta, \zeta\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$

Now (30) shows that (27) depends on (24) and (25). Hence, in this case, the four equations (24–27) are equivalent to two independent equations only, which is obvious since (13–16) are equivalent to two independent equations only.

Art. 4.—*To find the conditions which hold at every point on a Surface Locus of Unodal Lines.*

At such a point the tangent cone, whose equation is

$$[\xi, \xi](X - \xi)^2 + [\eta, \eta](Y - \eta)^2 + [\zeta, \zeta](Z - \zeta)^2$$
$$2[\eta, \zeta](Y - \eta)(Z - \zeta) + 2[\zeta, \xi](Z - \zeta)(X - \xi) + 2[\xi, \eta](X - \xi)(Y - \eta) = 0 \quad (33),$$

breaks up into two coincident planes.

Hence

$$\begin{aligned} &[\xi, \xi] : [\xi, \eta] : [\xi, \zeta] \\ =\ &[\eta, \xi] : [\eta, \eta] : [\eta, \zeta] \\ =\ &[\zeta, \xi] : [\zeta, \eta] : [\zeta, \zeta] \quad . \quad . \quad . \quad (34). \end{aligned}$$

Of the four equations (24)–(27), which are satisfied when there is a surface locus of binodal lines, it has already been shown that only two are independent. The same equations hold when there is a locus of unodal lines.

Multiply (24) by $[\eta, \xi]$, (25) by $[\xi, \xi]$, subtract and use (34). Then,

$$(\delta\alpha)\{[\eta, \xi][\xi, \alpha] - [\xi, \xi][\eta, \alpha]\} = 0 \quad . \quad . \quad . \quad (35),$$

therefore,

$$[\eta, \xi][\xi, \alpha] - [\xi, \xi][\eta, \alpha] = 0 \quad . \quad . \quad . \quad (36).$$

Similarly

$$[\zeta, \xi][\xi, \alpha] - [\xi, \xi][\zeta, \alpha] = 0 \quad . \quad . \quad . \quad (37).$$

By (34) and (36) it follows that (25) depends on (24).

By (34) and (37) it follows that (26) depends on (24).

By (34), (36), (37), it follows that, if the values of $\delta\xi : \delta\eta : \delta\zeta : \delta\alpha$ satisfying (24)–(27) are finite, then (27) depends on (24), and the following ratios hold:—

$$\begin{aligned} &[\xi, \xi] : [\xi, \eta] : [\xi, \zeta] : [\xi, \alpha] \\ =\ &[\eta, \xi] : [\eta, \eta] : [\eta, \zeta] : [\eta, \alpha] \\ =\ &[\zeta, \xi] : [\zeta, \eta] : [\zeta, \zeta] : [\zeta, \alpha] \\ =\ &[\alpha, \xi] : [\alpha, \eta] : [\alpha, \zeta] : [\alpha, \alpha] \quad . \quad . \quad . \quad (38). \end{aligned}$$

In this case, then, (24)–(27) are equivalent to one independent equation only.

It may be noticed that in the case in which

$$[\alpha, \xi] = 0, \ [\alpha, \eta] = 0, \ [\alpha, \zeta] = 0 \quad . \quad . \quad . \quad (39),$$

in order that (27) may be satisfied,

$$[\alpha, \alpha] = 0 \quad \ldots \ldots \ldots \ldots \quad (40).$$

And now (27) becomes an identical equation. It does not depend on (24).

Art. 5.—*Examination of the Discriminant* Δ, *and its Differential Coefficients, when a Surface Locus of Binodal Lines exists. Proof that* Δ *contains* B^2 *as a factor.*

Let ξ, η, ζ be a point on the binodal line on the surface (10).
Then when $x = \xi$, $y = \eta$, $z = \zeta$,

$$f_1 = 0, \quad \frac{Df_1}{Dx} = 0.$$

Hence by (6) and (7), when $x = \xi$, $y = \eta$, $z = \zeta$,

$$\Delta = 0, \quad \frac{\partial \Delta}{\partial x} = 0.$$

Similarly

$$\frac{\partial \Delta}{\partial y} = 0, \quad \frac{\partial \Delta}{\partial z} = 0.$$

Hence if $B = 0$ be the equation of the surface locus of binodal lines, Δ contains B^2 as a factor.

Example 2.—*Locus of Binodal Lines.*

Let the surfaces be

$$[\phi(x, y, z) - a]^2 + \chi(x, y, z)\,[\psi(x, y, z)]^2 = 0.$$

A. *The Discriminant.*

This is found by eliminating a between the above, and

$$-[2\,\phi(x, y, z) - a] = 0.$$

Hence the discriminant is $\chi(x, y, z)\,[\psi(x, y, z)]^2$.
Hence the locus of ultimate intersections is

$$\chi(x, y, z)\,[\psi(x, y, z)]^2 = 0.$$

B. *The Locus of Binodal Lines is* $\psi(x, y, z) = 0$.

Let ξ, η, ζ be any point on the surface $\psi(x, y, z) = 0$.

Take $a = \phi(\xi, \eta, \zeta)$ and consider the single surface

$$[\phi(x, y, z) - \phi(\xi, \eta, \zeta)]^2 + \chi(x, y, z)[\psi(x, y, z)]^2 = 0.$$

Put $x = \xi + X$, $y = \eta + Y$, $z = \zeta + Z$.

Then the lowest terms in X, Y, Z are

$$\left[X\frac{\partial\phi}{\partial\xi} + Y\frac{\partial\phi}{\partial\eta} + Z\frac{\partial\phi}{\partial\zeta}\right]^2 + \chi(\xi, \eta, \zeta)\left[X\frac{\partial\psi}{\partial\xi} + Y\frac{\partial\psi}{\partial\eta} + Z\frac{\partial\psi}{\partial\zeta}\right]^2.$$

This breaks up into two factors of the first degree in X, Y, Z.

Hence ξ, η, ζ is a binode on the surface considered. Now the only relation satisfied by ξ, η, ζ is $\psi(\xi, \eta, \zeta) = 0$.

Hence any point ξ, η, ζ, on the surface $\psi(x, y, z) = 0$, is a binode on

$$[\phi(x, y, z) - \phi(\xi, \eta, \zeta)]^2 + \chi(x, y, z)[\psi(x, y, z)]^2 = 0.$$

Hence every point of intersection of the surfaces

$$\psi(x, y, z) = 0,$$

and

$$[\phi(x, y, z) - a]^2 + \chi(x, y, z)[\psi(x, y, z)]^2 = 0,$$

is a binode on the latter surface.

The equations of the binodal line of this surface are, therefore,

$$\psi(x, y, z) = 0,$$
$$\phi(x, y, z) = a.$$

This accounts for the occurrence of the factor $[\psi(x, y, z)]^2$ in Δ.

C. *The Envelope Locus is* $\chi(x, y, z) = 0$.

This may be proved as in Example 1.

Art. 6.—*Examination of the Discriminant* Δ, *and its Differential Coefficients, when a Surface Locus of Unodal Lines exists. Proof that* Δ *contains* U^3 *as a factor.*

Differentiating (7) with regard to x and y,

$$\frac{\partial^2\Delta}{\partial x^2} = f_1\frac{\partial^2 Q}{\partial x^2} + 2\frac{\partial Q}{\partial x}\left(\frac{Df_1}{Dx} + \frac{Df_1}{Da_1}\frac{\partial a_1}{\partial x}\right) + Q\left(\frac{D^2f_1}{Dx^2} + \frac{D^2f_1}{Dx\,Da_1}\frac{\partial a_1}{\partial x}\right) \quad . \quad . \quad . \quad . \quad (41),$$

$$\frac{\partial^2\Delta}{\partial x\,\partial y} = f_1\frac{\partial^2 Q}{\partial x\,\partial y} + \frac{\partial Q}{\partial x}\left(\frac{Df_1}{Dy} + \frac{Df_1}{Da_1}\frac{\partial a_1}{\partial y}\right) + \frac{\partial Q}{\partial y}\frac{Df_1}{Dx} + Q\left(\frac{D^2f_1}{Dx\,Dy} + \frac{D^2f_1}{Dx\,Da_1}\frac{\partial a_1}{\partial y}\right) \quad (42).$$

To find $\partial a_1/\partial x$, $\partial a_1/\partial y$ it is necessary to use the equation

$$\frac{Df(x, y, z, a_1)}{Da_1} = 0 \quad \ldots\ldots\ldots\ldots \quad (43).$$

This gives

$$\frac{D^2f_1}{Dx\,Da_1} + \frac{D^2f_1}{Da_1^2}\frac{\partial a_1}{\partial x} = 0 \quad \ldots\ldots\ldots\ldots \quad (44),$$

$$\frac{D^2f_1}{Dy\,Da_1} + \frac{D^2f_1}{Da_1^2}\frac{\partial a_1}{\partial y} = 0 \quad \ldots\ldots\ldots\ldots \quad (45).$$

Now reserving the case, according to the remarks in the Abstract ('Proc. Roy. Soc., vol. 50, p. 180) and Art. 1, in which

$$\frac{D^2f_1}{Da_1^2} = 0 \quad \ldots\ldots\ldots\ldots \quad (46),$$

for further consideration, because, in this case, $\partial a_1/\partial x$, $\partial a_1/\partial y$, both become infinite or indeterminate, it follows that

$$\frac{\partial a_1}{\partial x} = -\frac{D^2f_1}{Dx\,Da_1}\Big/\frac{D^2f_1}{Da_1^2} \quad \ldots\ldots\ldots\ldots \quad (47),$$

$$\frac{\partial a_1}{\partial y} = -\frac{D^2f_1}{Dy\,Da_1}\Big/\frac{D^2f_1}{Da_1^2} \quad \ldots\ldots\ldots\ldots \quad (48).$$

Hence

$$\frac{\partial^2\Delta}{\partial x^2} = \frac{\partial^2 Q}{\partial x^2}f_1 + 2\frac{\partial Q}{\partial x}\frac{Df_1}{Dx} + Q\left\{\frac{D^2f_1}{Dx^2}\frac{D^2f_1}{Da_1^2} - \left(\frac{D^2f_1}{Dx\,Da_1}\right)^2\right\}\Big/\frac{D^2f_1}{Da_1^2} \quad \ldots\ldots \quad (49),$$

$$\frac{\partial^2\Delta}{\partial x\,\partial y} = \frac{\partial^2 Q}{\partial x\,\partial y}f_1 + \frac{\partial Q}{\partial x}\frac{Df_1}{Dy} + \frac{\partial Q}{\partial y}\frac{Df_1}{Dx} + Q\left\{\frac{D^2f_1}{Dx\,Dy}\frac{D^2f_1}{Da_1^2} - \frac{D^2f_1}{Dx\,Da_1}\frac{D^2f_1}{Dy\,Da_1}\right\}\Big/\frac{D^2f_1}{Da_1^2} \,. \quad (50).$$

Hence, if ξ, η, ζ be a unode on the surface (10), and a_1 become equal to α when $x = \xi$, $y = \eta$, $z = \zeta$, then by means of (38),

$$\frac{\partial^2\Delta}{\partial x^2} = 0, \qquad \frac{\partial^2\Delta}{\partial x\,\partial y} = 0,$$

when $x = \xi$, $y = \eta$, $z = \zeta$.

Similarly all the other second differential coefficients of Δ with regard to x, y, z vanish when $x = \xi$, $y = \eta$, $z = \zeta$.

Hence, if $U = 0$ be the equation of the surface locus of unodal lines, Δ contains U^3 as a factor.

Example 3.—*Locus of Unodal Lines.*

Let the surfaces be

$$[\phi(x, y, z) - a]^2 + [\chi(x, y, z)]^3 = 0.$$

(A.) *The Discriminant.*

The discriminant is found by eliminating a between the above and

$$-2[\phi(x, y, z) - a] = 0.$$

Hence it is

$$[\chi(x, y, z)]^3.$$

Hence the locus of ultimate intersections is

$$[\chi(x, y, z)]^3 = 0.$$

(B.) *The Locus of Unodal Lines is* $\chi(x, y, z) = 0$.

Let ξ, η, ζ be any point on the locus $\chi(x, y, z) = 0$.
Let $a = \phi(\xi, \eta, \zeta)$, and consider the single surface

$$[\phi(x, y, z) - \phi(\xi, \eta, \zeta)]^2 + [\chi(x, y, z)]^3 = 0.$$

Put $x = \xi + X$, $y = \eta + Y$, $z = \zeta + Z$; then the lowest terms in X, Y, Z are

$$\left[X\frac{\partial\phi}{\partial\xi} + Y\frac{\partial\phi}{\partial\eta} + Z\frac{\partial\phi}{\partial\zeta}\right]^2.$$

Hence ξ, η, ζ is a unode on the surface

$$[\phi(x, y, z) - \phi(\xi, \eta, \zeta)]^2 + [\chi(x, y, z)]^3 = 0.$$

Hence $\chi(x, y, z) = 0$ is the locus of unodal lines on the surfaces

$$[\phi(x, y, z) - a]^2 + [\chi(x, y, z)]^3 = 0.$$

The unodal line on any one of the surfaces is given by the equations

$$\chi(x, y, z) = 0,$$
$$\phi(x, y, z) = a.$$

This accounts for the occurrence of the factor $[\chi(x, y, z)]^3$ in the discriminant.

SECTION II. (Arts. 7–9).—CONSIDERATION OF THE CASES RESERVED IN WHICH TWO ROOTS OF THE EQUATION $Df/D\alpha = 0$ BECOME EQUAL AT ANY POINT ON THE LOCUS OF ULTIMATE INTERSECTIONS.

Art. 7.—*Consideration of the exceptional case of the Envelope Locus, in which two consecutive characteristics coincide.*

(A.) It will be shown that this is the case reserved in Art. 1, viz., where $D^2f_1/D\alpha_1^2 = 0$. The geometrical meaning of the condition will first of all be determined.

The surface

$$f(x, y, z, \alpha) = 0$$

intersects the surface

$$f(x, y, z, \alpha + \delta\alpha) = 0,$$

where $\delta\alpha$ is indefinitely small in the curve whose equations are

$$f(x, y, z, \alpha) = 0\,,$$

$$\frac{Df(x, y, z, \alpha)}{D\alpha} = 0\,.$$

This curve is called a characteristic. The equations of the next characteristic are obtained by changing α into $\alpha + \delta\alpha$ in the above. Hence they are

$$f(x, y, z, \alpha) + (\delta\alpha)\frac{Df(x, y, z, \alpha)}{D\alpha} = 0\,,$$

$$\frac{Df(x, y, z, \alpha)}{D\alpha} + (\delta\alpha)\frac{D^2f(x, y, z, \alpha)}{D\alpha^2} = 0\,.$$

Now, if the two consecutive characteristics coincide,

$$f(x, y, z, \alpha) = 0\,,\quad \frac{Df(x, y, z, \alpha)}{D\alpha} = 0\,,\quad \frac{D^2f(x, y, z, \alpha)}{D\alpha^2} = 0\,,$$

at every point of the coinciding characteristics.

Hence, the characteristic counts three times over as an intersection of the envelope and the surface, instead of twice as in the ordinary case.

(B.) It is now necessary to repeat the investigation in the case in which equation (2) has equal roots when $x = \xi$, $y = \eta$, $z = \zeta$, the co-ordinates of a point on the locus of ultimate intersections.

In this case, Δ may be written

$$Rf(x, y, z, a_1) f(x, y, z, a_2) = Rf_1 f_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

where a_1, a_2 are the roots of (2) which become equal when $x = \xi$, $y = \eta$, $z = \zeta$.

Therefore,

$$\frac{\partial \Delta}{\partial x} = \frac{\partial R}{\partial x} f_1 f_2 + R f_2 \left(\frac{Df_1}{Dx} + \frac{Df_1}{Da_1} \frac{\partial a_1}{\partial x} \right) + R f_1 \left(\frac{Df_2}{Dx} + \frac{Df_2}{Da_2} \frac{\partial a_2}{\partial x} \right) \quad . \quad . \quad (52).$$

Now if it be assumed (see immediately below, under C) that the terms $f_2 \frac{Df_1}{Da_1} \frac{\partial a_1}{\partial x}$, $f_1 \frac{Df_2}{Da_2} \frac{\partial a_2}{\partial x}$ vanish, then when $x = \xi$, $y = \eta$, $z = \zeta$, it follows that $a_1 = a_2 = \alpha$, and, therefore, $f_1 = 0$, $f_2 = 0$, and $\frac{\partial \Delta}{\partial x} = 0$.

Similarly, $\frac{\partial \Delta}{\partial y} = 0$, $\frac{\partial \Delta}{\partial z} = 0$.

Therefore, Δ contains E^2 as a factor.

(C.) *Examination of the term* $f_2 \frac{Df_1}{Da_1} \frac{\partial a_1}{\partial x}$,

Taking $\partial a_1 / \partial x$ from (47) this term becomes

$$- f_2 \frac{Df_1}{Da_1} \frac{D^2 f_1}{Da_1 Dx} \bigg/ \frac{D^2 f_1}{Da_1^{\,2}}.$$

Now f is of the form

$$A (a - \alpha_1)(a - \alpha_2)(a - \alpha_3),$$

where α_1, α_2, α_3 all become equal to the same thing as a_1, a_2 when $x = \xi$, $y = \eta$, $z = \zeta$. Hence, taking as infinitesimal of the first order the difference in the values of the parameter a at the points ξ, η, ζ and $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$, it follows that f_2 is of the third order of small quantities, Df_1/Da_1 of the second, and $D^2 f_1/Da_1^{\,2}$ of the first. Hence, assuming that $D^2 f_1/Da_1 Dx$ is not infinite, it follows that the term under investigation is of the fourth order, and therefore vanishes ultimately.

Example 4.—*Envelope Locus when two consecutive Characteristics coincide.*

Let the surfaces be

$$\phi(x, y, z) + [\psi(x, y, z) - a]^3 = 0.$$

(A). *The Discriminant.*

The discriminant is found by eliminating a between the above and

$$- 3 [\psi(x, y, z) - a]^2 = 0,$$

Hence it is $[\phi(x, y, z)]^3$.

Hence the locus of ultimate intersections is

$$[\phi(x, y, z)]^2 = 0.$$

(B). *The Envelope Locus such that two consecutive Characteristics coincide is* $\phi(x, y, z) = 0$.

Let ξ, η, ζ be any point on $\phi(x, y, z) = 0$.

Take $a = \psi(\xi, \eta, \zeta)$, and consider the single surface

$$\phi(x, y, z) + [\psi(x, y, z) - \psi(\xi, \eta, \zeta)]^3 = 0.$$

Put $x = \xi + X$, $y = \eta + Y$, $z = \zeta + Z$: then the lowest terms in X, Y, Z are

$$X\frac{\partial\phi}{\partial\xi} + Y\frac{\partial\phi}{\partial\eta} + Z\frac{\partial\phi}{\partial\zeta}.$$

Hence the tangent plane to the surface at ξ, η, ζ is also the tangent plane to $\phi(x, y, z) = 0$.

Hence $\phi(x, y, z) = 0$ is the envelope, and the equations of the line of contact are $\phi(x, y, z) = 0$, $\psi(x, y, z) = \psi(\xi, \eta, \zeta)$.

Now the equations corresponding to $f = 0$, $Df/Da = 0$, $D^2f/Da^2 = 0$ are

$$\begin{aligned} \phi(x, y, z) + \quad [\psi(x, y, z) - a]^3 &= 0, \\ -3[\psi(x, y, z) - a]^2 &= 0, \\ 6[\psi(x, y, z) - a] &= 0. \end{aligned}$$

These are all satisfied by the coordinates of any point on the line of contact. Hence two consecutive characteristics coincide. This accounts for the factor $[\phi(x, y, z)]^2$ in the discriminant.

Art. 8.—*Consideration of Loci of Binodal Lines, which are also Envelopes.*

(A.) It will be shown that this is the case reserved in Art. 1, viz., where $D^2f_1/Da_1^2 = 0$.

The equation of the biplanes is

$$[\xi, \xi](X - \xi)^2 + [\eta, \eta](Y - \eta)^2 + [\zeta, \zeta](Z - \zeta)^2$$
$$+ 2[\eta, \zeta](Y - \eta)(Z - \zeta) + 2[\zeta, \xi](Z - \zeta)(X - \xi) + 2[\xi, \eta](X - \xi)(Y - \eta) = 0 \quad (53).$$

This breaks up by (29) into the two planes

$$[\xi, \xi]\{[\eta, \xi](X-\xi)+[\eta, \eta](Y-\eta)+[\eta, \zeta](Z-\zeta)\}$$
$$+\{-[\xi, \eta] \pm \sqrt{[\xi, \eta]^2-[\xi, \xi][\eta, \eta]}\}\{[\xi, \xi](X-\xi)+[\xi, \eta](Y-\eta)$$
$$+[\xi, \zeta](Z-\zeta)\}=0 \quad . \quad (54).$$

Now since equation (27) depends on (24) and (25), therefore

$$\begin{vmatrix} [\xi, \xi] & [\xi, \eta] & [\xi, \alpha] \\ [\eta, \xi] & [\eta, \eta] & [\eta, \alpha] \\ [\alpha, \xi] & [\alpha, \eta] & [\alpha, \alpha] \end{vmatrix}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55).$$

Putting $[\alpha, \alpha]=0$, this becomes

$$[\alpha, \eta]^2[\xi, \xi]-2[\alpha, \eta][\alpha, \xi][\xi, \eta]+[\alpha, \xi]^2[\eta, \eta]=0 \quad . \quad . \quad . \quad (56).$$

Therefore

$$[\alpha, \eta]/[\alpha, \xi]=\{[\xi, \eta] \pm \sqrt{[\xi, \eta]^2-[\xi, \xi][\eta, \eta]}\}/[\xi, \xi] \quad . \quad . \quad . \quad (57).$$

Hence the equation of *one* of the biplanes is

$$[\alpha, \xi]\{[\xi, \eta](X-\xi)+[\eta, \eta](Y-\eta)+[\zeta, \eta](Z-\zeta)\}$$
$$-[\alpha, \eta]\{[\xi, \xi](X-\xi)+[\xi, \eta](Y-\eta)+[\xi, \zeta](Z-\zeta)\}=0 \quad . \quad . \quad (58).$$

Now if $\xi+\delta\xi, \eta+\delta\eta, \zeta+\delta\zeta$ be a point on the locus of binodal lines near to ξ, η, ζ, it follows by (24) and (25) that

$$[\alpha, \xi]\{[\xi, \eta](\delta\xi)+[\eta, \eta](\delta\eta)+[\zeta, \eta](\delta\zeta)\}$$
$$-[\alpha, \eta]\{[\xi, \xi](\delta\xi)+[\xi, \eta](\delta\eta)+[\xi, \zeta](\delta\zeta)\}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59).$$

Hence the tangent plane to the locus of binodal lines takes the same form as (58).

Hence the tangent plane to the locus of binodal lines is the same as one of the biplanes.

Hence the locus of binodal lines is also an envelope.

(B.) The converse proposition, viz., that if the locus of binodal lines be also an envelope, then $[\alpha, \alpha]=0$, will now be proved.

As before, the equations of the biplanes are given by (54), and the tangent plane to the locus of binodal lines takes the same form as (58). If, then, the locus of binodal lines be also an envelope

$$\{-[\xi, \eta] \pm \sqrt{[\xi, \eta]^2-[\xi, \xi][\eta, \eta]}\}/[\xi, \xi]=-[\alpha, \eta]/[\alpha, \xi] \quad . \quad . \quad (60),$$

therefore

$$[\alpha, \eta]^2[\xi, \xi]-2[\alpha, \eta][\alpha, \xi][\xi, \eta]+[\alpha, \xi]^2[\eta, \eta]=0 \quad . \quad . \quad . \quad (61).$$

The equation (61) is the same as (56).

Comparing it with (55), which holds when a locus of binodal lines exists, it follows that

$$\{[\xi, \xi][\eta, \eta] - [\xi, \eta]^2\}[\alpha, \alpha] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62).$$

Hence $[\alpha, \alpha] = 0$, or

$$[\xi, \xi][\eta, \eta] - [\xi, \eta]^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63).$$

If (63) hold, then by (61)

$$[\alpha, \eta][\xi, \xi] = [\alpha, \xi][\xi, \eta],$$

therefore

$$[\xi, \xi]/[\xi, \eta] = [\xi, \eta]/[\eta, \eta] = [\alpha, \xi]/[\alpha, \eta].$$

Making use of these with (24) and (25), it follows that each of these fractions is equal to $[\xi, \zeta]/[\eta, \zeta]$.

Hence (24) and (25) are equivalent to one equation only.

But it was shown that (26) and (27) depend on (24) and (25). Hence (25), (26), (27) all depend on (24).

Hence the ratios (38) hold, and therefore there is a locus of unodal lines. But this is not the case under consideration, for it is supposed that there is a locus of binodal, not unodal, lines.

Hence (63) is not satisfied, and, therefore, $[\alpha, \alpha] = 0$.

(C.) In this case the values of $\partial a_1/\partial x$, $\partial a_1/\partial y$, given by (47) and (48), are really infinite, for $D^2 f_1/Da_1\,Dx$ does not vanish necessarily, but $D^2 f_1/Da_1^2 = 0$.

Consequently the differential coefficients of Δ require further examination.

Now

$$f_2 \frac{D^2 f_1}{Da_1\,Dx} \frac{\partial a_1}{\partial x} = -f_2 \left(\frac{D^2 f_1}{Da_1\,Dx}\right)^2 \Big/ \frac{D^2 f_1}{Da_1^2}.$$

Since $D^2 f_1/Da_1\,Dx$ does not necessarily vanish, it must be shown that $f_2 \Big/ \frac{D^2 f_1}{Da_1^2} = 0$ at points on the locus of binodal lines; *i.e.*,

$$f(x, y, z, a_2) \Big/ \frac{D^2 f(x, y, z, a_1)}{Da_1^2} = 0,$$

when $x = \xi$, $y = \eta$, $z = \zeta$, the coordinates of any point on the locus of binodal lines.

Now a_1, a_2 are the roots of

$$\frac{Df(x, y, z, a)}{Da} = 0,$$

which become equal when $x = \xi$, $y = \eta$, $z = \zeta$.

In this case $f(x, y, z, a) = 0$ is an equation for a, such that three values become equal when $x = \xi$, $y = \eta$, $z = \zeta$. (They become equal to the same value as a_1, a_2.)

Put, therefore,

$$f = \mathrm{R}\,(a - \alpha_1)(a - \alpha_2)(a - \alpha_3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64),$$

where α_1, α_2, α_3 become equal when $x = \xi$, $y = \eta$, $z = \zeta$.

Therefore

$$\frac{\mathrm{D}f}{\mathrm{D}a} = \frac{\mathrm{DR}}{\mathrm{D}a}(a - \alpha_1)(a - \alpha_2)(a - \alpha_3)$$
$$+ \mathrm{R}\,\{(a - \alpha_2)(a - \alpha_3) + (a - \alpha_3)(a - \alpha_1) + (a - \alpha_1)(a - \alpha_2)\} \quad . \quad (65),$$

$$\frac{\mathrm{D}^2}{\mathrm{D}a^2} = \frac{\mathrm{D}^2\mathrm{R}}{\mathrm{D}a^2}(a - \alpha_1)(a - \alpha_2)(a - \alpha_3)$$
$$+ 2\frac{\mathrm{DR}}{\mathrm{D}a}\{(a - \alpha_2)(a - \alpha_3) + (a - \alpha_3)(a - \alpha_1) + (a - \alpha_1)(a - \alpha_2)\}$$
$$+ 2\mathrm{R}\,\{(a - \alpha_1) + (a - \alpha_2) + (a - \alpha_3)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66).$$

Now, at a point on the locus of binodal lines, the two equal values of a which make $\mathrm{D}f/\mathrm{D}a = 0$, become equal to the same thing as α_1, α_2, α_3.

Hence $f(x, y, z, a_2) = (\mathrm{R}')(a_2 - \alpha_1)(a_2 - \alpha_2)(a_2 - \alpha_3)$, where R' is what R becomes, when a is changed into a_2.

Hence $f(x, y, z, a_2)$ is of the third order of small quantities; but $\mathrm{D}^2f(x, y, z, a_1)/\mathrm{D}a_1^2$ is of the first order, for the most important term in it is

$$2\,(\mathrm{R}')\,[(a_1 - \alpha_1) + (a_1 - \alpha_2) + (a_1 - \alpha_3)].$$

Hence

$$f(x, y, z, a_2)\Big/\frac{\mathrm{D}^2f(x, y, z, a_1)}{\mathrm{D}a_1^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

at points on the locus of binodal lines.

In like manner

$$f(x, y, z, a_2)\Big/\left\{\frac{\mathrm{D}^2f(x, y, z, a_1)}{\mathrm{D}a_1^2}\right\}^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (68),$$

but

$$f(x, y, z, a_2)\Big/\left\{\frac{\mathrm{D}^2f(x, y, z, a_1)}{\mathrm{D}a_1^2}\right\}^3 \neq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

at points on the locus of binodal lines.

(D). It should be noticed that in the preceding section (C), the infinitesimal of the first order is the increment in the value of α, a root of $Df(x, y, z, \alpha)/D\alpha = 0$, when ξ, η, ζ receive increments $\delta\xi, \delta\eta, \delta\zeta$ respectively; and in particular that it is not of the same order as $\delta\xi, \delta\eta, \delta\zeta$.

For if $\delta\alpha$ be the increment in the value α, then

$$\begin{aligned}
&[\alpha, \xi](\delta\xi) + [\alpha, \eta](\delta\eta) + [\alpha, \zeta](\delta\zeta) + [\alpha, \alpha](\delta\alpha) \\
&+ \tfrac{1}{2}\left[\begin{array}{l} [\alpha, \xi, \xi](\delta\xi)^2 + [\alpha, \eta, \eta](\delta\eta)^2 + [\alpha, \zeta, \zeta](\delta\zeta)^2 + [\alpha, \alpha, \alpha](\delta\alpha)^2 \\ + 2[\alpha, \eta, \zeta](\delta\eta)(\delta\zeta) + 2[\alpha, \zeta, \xi](\delta\zeta)(\delta\xi) + 2[\alpha, \xi, \eta](\delta\xi)(\delta\eta) \\ + 2[\alpha, \alpha, \xi](\delta\alpha)(\delta\xi) + 2[\alpha, \alpha, \eta](\delta\alpha)(\delta\eta) + 2[\alpha, \alpha, \zeta](\delta\alpha)(\delta\zeta) \end{array}\right] \\
&+ \ldots \ldots = 0.
\end{aligned}$$

Now because $[\alpha, \alpha] = 0$, this equation can be written in the form

$$u_1 + u_2 + 2v_1(\delta\alpha) + v_0(\delta\alpha)^2 = 0,$$

where the suffixes denote the order of the terms, when $(\delta\xi), (\delta\eta), (\delta\zeta)$ are taken to be of the first order.

Hence, if ϵ denote an infinitely small quantity of the first order in $(\delta\xi), (\delta\eta), (\delta\zeta)$, then $\delta\alpha$ is of the order $\epsilon^{1/2}$.

And now $f(x, y, z, a_2)$, when $x = \xi + \delta\xi$, $y = \eta + \delta\eta$, $z = \zeta + \delta\zeta$, and $a_2 = \alpha + \delta\alpha$, becomes

$$\begin{aligned}
&f(\xi, \eta, \zeta, \alpha) + [\xi](\delta\xi) + [\eta](\delta\eta) + [\zeta](\delta\zeta) + [\alpha](\delta\alpha) \\
&+ \tfrac{1}{2}\left[\begin{array}{l} [\xi, \xi](\delta\xi)^2 + [\eta, \eta](\delta\eta)^2 + [\zeta, \zeta](\delta\zeta)^2 + [\alpha, \alpha](\delta\alpha)^2 \\ + 2[\eta, \zeta](\delta\eta)(\delta\zeta) + 2[\zeta, \xi](\delta\zeta)(\delta\xi) + 2[\xi, \eta](\delta\xi)(\delta\eta) \\ + 2[\alpha, \xi](\delta\alpha)(\delta\xi) + 2[\alpha, \eta](\delta\alpha)(\delta\eta) + 2[\alpha, \zeta](\delta\alpha)(\delta\zeta) \end{array}\right] \\
&+ \ldots \\
&= \tfrac{1}{2}\left[\begin{array}{l} [\xi, \xi](\delta\xi)^2 + [\eta, \eta](\delta\eta)^2 + [\zeta, \zeta](\delta\zeta)^2 \\ + 2[\eta\zeta](\delta\eta)(\delta\zeta) + 2[\zeta, \xi](\delta\zeta)(\delta\xi) + 2[\xi, \eta](\delta\xi)(\delta\eta) \\ + 2[\alpha, \xi](\delta\alpha)(\delta\xi) + 2[\alpha, \eta](\delta\alpha)(\delta\eta) + 2[\alpha, \zeta](\delta\alpha)(\delta\zeta) \end{array}\right] \\
&+ \ldots
\end{aligned}$$

Hence $f(x, y, z, a_2)$ is of the order $\epsilon^{3/2}$ when $x = \xi + \delta\xi$, $y = \eta + \delta\eta$, $z = \zeta + \delta\zeta$.

In like manner, when $x = \xi + \delta\xi$, $y = \eta + \delta\eta$, $z = \zeta + \delta\zeta$, $a_1 = \alpha + \delta\alpha'$,

$$\frac{D^2 f(x, y, z, a_1)}{Da_1^2} = [\alpha, \alpha]$$
$$+ [\alpha, \alpha, \xi](\delta\xi) + [\alpha, \alpha, \eta](\delta\eta) + [\alpha, \alpha, \zeta](\delta\zeta) + [\alpha, \alpha, \alpha_i](\delta\alpha')$$
$$+ \ldots$$

Now $\delta\alpha'$ can be shown to be of the order $\epsilon^{1/2}$, in the same way that $\delta\alpha$ was shown to be of this order.

Hence $D^2 f(x, y, z, a_1)/Da_1^2$ is of the order $\epsilon^{1/2}$.

Hence the same results as those given in (67), (68), (69) follow.

(E.) *Examination of the Differential Coefficients of* Δ.

Differentiating (52) with regard to x,

$$\frac{\partial^2 \Delta}{\partial x^2} = f_1 f_2 \frac{\partial^2 R}{\partial x^2} + 2\frac{\partial R}{\partial x}\left(f_2 \frac{Df_1}{Dx} + f_1 \frac{Df_2}{Dx}\right)$$
$$+ R\left[\begin{array}{l} f_2\left(\dfrac{D^2 f_1}{Dx^2} + \dfrac{D^2 f_1}{Dx\, Da_1}\dfrac{\partial a_1}{\partial x}\right) \\ + 2\dfrac{Df_1}{Dx}\dfrac{Df_2}{Dx} \\ + f_1\left(\dfrac{D^2 f_2}{Dx^2} + \dfrac{D^2 f_2}{Dx\, Da_2}\dfrac{\partial a_2}{\partial x}\right)\end{array}\right] \quad \ldots \ldots \quad (70).$$

Hence the term

$$f_2 \frac{D^2 f_1}{Dx\, Da_1}\frac{\partial a_1}{\partial x} = -f_2\left(\frac{D^2 f_1}{Dx\, Da_1}\right)^2 \Big/ \frac{D^2 f_1}{Da_1^2}$$

vanishes by (67) at points on the locus of binodal lines.

Hence $\partial^2\Delta/\partial x^2 = 0$ at points on the locus of binodal lines, if it be assumed that $D^2 f_1/Dx\, Da_1$ is finite.

This assumption can be made if a_1 be finite.

Again, differentiating (70) with regard to x,

$$\frac{\partial^3 \Delta}{\partial x^3} = \frac{\partial^3 R}{\partial x^3} f_1 f_2 + 3\frac{\partial R}{\partial x}\left(f_2 \frac{Df_1}{Dx} + f_1 \frac{Df_2}{Dx}\right)$$
$$+ 3R\left[f_2\left(\frac{D^2 f_1}{Dx^2} + \frac{D^2 f_1}{Dx\, Da_1}\frac{\partial a_1}{\partial x}\right) + 2\frac{Df_1}{Dx}\frac{Df_2}{Dx} + f_1\left(\frac{D^2 f_2}{Dx^2} + \frac{D^2 f_2}{Dx\, Da_2}\frac{\partial a_2}{\partial x}\right)\right]$$
$$+ R\left[\begin{array}{l} f_2 \dfrac{\partial}{\partial x}\left(\dfrac{D^2 f_1}{Dx^2} + \dfrac{D^2 f_1}{Dx\, Da_1}\dfrac{\partial a_1}{\partial x}\right) + 3\dfrac{Df_2}{Dx}\left(\dfrac{D^2 f_1}{Dx^2} + \dfrac{D^2 f_1}{Dx\, Da_1}\dfrac{\partial a_1}{\partial x}\right) \\ + 3\dfrac{Df_1}{Dx}\left(\dfrac{D^2 f_2}{Dx^2} + \dfrac{D^2 f_2}{Dx\, Da_2}\dfrac{\partial a_2}{\partial x}\right) + f_1 \dfrac{\partial}{\partial x}\left(\dfrac{D^2 f_2}{Dx^2} + \dfrac{D^2 f_2}{Dx\, Da_2}\dfrac{\partial a_2}{\partial x}\right)\end{array}\right] \quad . . \quad (71).$$

In this case the infinite quantity $\partial a_1/\partial x$ occurs in the term

$$f_2 \frac{\partial}{\partial x}\left(\frac{D^2 f_1}{Dx^2} + \frac{D^2 f_1}{Dx\,Da_1}\frac{\partial a_1}{\partial x}\right)$$

$$= f_2 \frac{\partial}{\partial x}\left\{\left(\frac{D^2 f_1}{Dx^2}\frac{D^2 f_1}{Da_1^2} - \left(\frac{D^2 f_1}{Dx\,Da_1}\right)^2\right)\Big/\frac{D^2 f_1}{Da_1^2}\right\}$$

$$= f_2 \left[\begin{array}{l}\dfrac{D}{Dx}\left\{\left(\dfrac{D^2 f_1}{Dx^2}\dfrac{D^2 f_1}{Da_1^2} - \left(\dfrac{D^2 f_1}{Dx\,Da_1}\right)^2\right)\Big/\dfrac{D^2 f_1}{Da_1^2}\right\} \\ - \dfrac{\dfrac{D^2 f_1}{Dx\,Da_1}}{\dfrac{D^2 f_1}{Da_1^2}}\dfrac{D}{Da_1}\left\{\left(\dfrac{D^2 f_1}{Dx^2}\dfrac{D^2 f_1}{Da_1^2} - \left(\dfrac{D^2 f_1}{Dx\,Da_1}\right)^2\right)\Big/\dfrac{D^2 f_1}{Da_1^2}\right\}\end{array}\right]$$

$$= f_2 \frac{D^3 f_1}{Dx^3} - 3\frac{f_2}{\dfrac{D^2 f_1}{Da_1^2}}\frac{D^2 f_1}{Dx\,Da_1}\frac{D^3 f_1}{Dx^2\,Da_1}$$

$$+ 3\frac{f_2}{\left(\dfrac{D^2 f_1}{Da_1^2}\right)^2}\left(\frac{D^2 f_1}{Dx\,Da_1}\right)^2\frac{D^3 f_1}{Dx\,Da_1^2} - \frac{f_2}{\left(\dfrac{D^2 f_1}{Da_1^2}\right)^3}\left(\frac{D^2 f_1}{Dx\,Da_1}\right)^3\frac{D^3 f_1}{Da_1^3}.$$

Now it has been shown in (67)–(69) that $f_2\Big/\dfrac{D^2 f_1}{Da_1^2}$ and $f_2\Big/\left(\dfrac{D^2 f_1}{Da_1^2}\right)^2$ both vanish, but $f\Big/\left(\dfrac{D^2 f_1}{Da_1^2}\right)^3$ does not vanish at points on the locus of binodal lines.

Hence $f_2 \dfrac{\partial}{\partial x}\left(\dfrac{D^2 f_1}{Dx^2} + \dfrac{D^2 f_1}{Dx\,Da_1}\dfrac{\partial a_1}{\partial x}\right)$ does not vanish.

Similarly $f_1 \dfrac{\partial}{\partial x}\left(\dfrac{D^2 f_2}{Dx^2} + \dfrac{D^2 f_2}{Dx\,Da_2}\dfrac{\partial a_2}{\partial x}\right)$ does not vanish.

The order of the term

$$\frac{Df_2}{Dx}\frac{D^2 f_1}{Dx\,Da_1}\frac{\partial a_1}{\partial x}$$

cannot be determined in a perfectly general way, for although Df_2/Dx vanishes, yet it contains $\partial \alpha_1/\partial x$, $\partial \alpha_2/\partial x$, $\partial \alpha_3/\partial x$, which may be infinite, since α_1, α_2, α_3 are irrational functions of the coordinates.

These results point to the conclusion that $\partial^3\Delta/\partial x^3$ does not vanish at all points on the locus of binodal lines. This is readily proved in particular cases. (See Example 5 below.)

Hence, at points on the locus of binodal lines, $\Delta = 0$, $\partial\Delta/\partial x = 0$, $\partial^2\Delta/\partial x^2 = 0$; but $\partial^3\Delta/\partial x^3 \neq 0$.

Hence, if $B = 0$ be the equation of the locus of binodal lines, when that locus is also an envelope, Δ contains B^3 as a factor.

Example 5.—*Locus of Binodal Lines which is also an Envelope.*

Let the surfaces be

$$[\phi(x, y, z)]^2 + \phi(x, y, z)[\psi(x, y, z) - a] + [\psi(x, y, z) - a]^3 = 0.$$

(A.) *The Discriminant.*

The discriminant is found by eliminating a between the above and

$$\phi(x, y, z) + 3[\psi(x, y, z) - a]^2 = 0.$$

The last equation gives

$$\psi(x, y, z) - a = \pm\sqrt{\{-\tfrac{1}{3}\phi(x, y, z)\}}.$$

Hence the eliminant is

$$\{[\phi(x, y, z)]^2 + \tfrac{2}{3}\phi(x, y, z)\sqrt{-\tfrac{1}{3}\phi(x, y, z)}\}$$
$$\times\{[\phi(x, y, z)]^2 - \tfrac{2}{3}\phi(x, y, z)\sqrt{-\tfrac{1}{3}\phi(x, y, z)}\}$$
$$= [\phi(x, y, z)]^3[\phi(x, y, z) + \tfrac{4}{2}$$

Hence the locus of ultimate intersections is

$$[\phi(x, y, z)]^3[\phi(x, y, z) + \tfrac{4}{27}] = 0.$$

(B.) *The locus of Binodal Lines (which is also an Envelope) is* $\phi(x, y, z) = 0$.

For let ξ, η, ζ be any point on $\phi(x, y, z) = 0$.

Take $a = \psi(\xi, \eta, \zeta)$, and consider the single surface

$$[\phi(x, y, z)]^2 + \phi(x, y, z)[\psi(x, y, z) - \psi(\xi, \eta, \zeta)] + [\psi(x, y, z) - \psi(\xi, \eta, \zeta)]^3 = 0.$$

Put $x = \xi + X$, $y = \eta + Y$, $z = \zeta + Z$; then the lowest terms in X, Y, Z are

$$\left(X\frac{\partial\phi}{\partial\xi} + Y\frac{\partial\phi}{\partial\eta} + Z\frac{\partial\phi}{\partial\zeta}\right)\left\{\left(X\frac{\partial\phi}{\partial\xi} + Y\frac{\partial\phi}{\partial\eta} + Z\frac{\partial\phi}{\partial\zeta}\right) + \left(X\frac{\partial\psi}{\partial\xi} + Y\frac{\partial\psi}{\partial\eta} + Z\frac{\partial\psi}{\partial\zeta}\right)\right\}.$$

Hence the origin is a binode.

Hence $\phi(x, y, z) = 0$ is the locus of binodal lines.

Further, because the biplanes are, when the origin is at the binode,

$$X\frac{\partial\phi}{\partial\xi} + Y\frac{\partial\phi}{\partial\eta} + Z\frac{\partial\phi}{\partial\zeta} = 0,$$

$$\left(X\frac{\partial\phi}{\partial\xi} + Y\frac{\partial\phi}{\partial\eta} + Z\frac{\partial\phi}{\partial\zeta}\right) + \left(X\frac{\partial\psi}{\partial\xi} + Y\frac{\partial\psi}{\partial\eta} + Z\frac{\partial\psi}{\partial\zeta}\right) = 0.$$

The first of these touches the locus of binodal lines. Hence the locus of binodal lines is also an envelope.

Hence the factor $[\phi(x, y, z)]^3$ of the discriminant is accounted for.

(C.) *The Surface* $\phi(x, y, z) + \frac{4}{27} = 0$ *is an Envelope.*

For seeking its intersection with

$$[\phi(x, y, z)]^2 + \phi(x, y, z,)[\psi(x, y, z) - a] + [\psi(x, y, z) - a]^3 = 0,$$

it follows that

$$[\psi(x, y, z) - a]^3 - \tfrac{4}{27}[\psi(x, y,z) - a] + \tfrac{16}{729} = 0.$$

Put

$$\psi(x, y, z) - a = \eta/9,$$

therefore

$$\eta^3 - 12\eta + 16 = 0,$$

therefore

$$(\eta - 2)^2(\eta + 4) = 0,$$

$$i.e., \{9[\psi(x, y, z) - a] - 2\}^2\{9[\psi(x, y, z) - a] + 4\} = 0.$$

Consider now any point ξ, η, ζ on the surface

$$[\phi(x, y, z)]^2 + \phi(x, y, z)[\psi(x, y, z) - a] + [\psi(x, y, z) - a]^3 = 0,$$

for which

$$\phi(\xi, \eta, \zeta) + \tfrac{4}{27} = 0,$$

and

$$\psi(\xi, \eta, \zeta) - a - \tfrac{2}{9} = 0.$$

The equation of the tangent plane at such a point is

$$[2\phi(\xi, \eta, \zeta) + \psi(\xi, \eta, \zeta) - a]\left[(X - \xi)\frac{\partial\phi}{\partial\xi} + (Y - \eta)\frac{\partial\phi}{\partial\eta} + (Z - \zeta)\frac{\partial\phi}{\partial\zeta}\right]$$
$$+ \{\phi(\xi, \eta, \zeta) + 3[\psi(\xi, \eta, \zeta) - a]^2\}\left[(X - \xi)\frac{\partial\psi}{\partial\xi} + (Y - \eta)\frac{\partial\psi}{\partial\eta} + (Z - \zeta)\frac{\partial\psi}{\partial\zeta}\right] = 0.$$

This reduces to

$$(X - \xi)\frac{\partial\phi}{\partial\xi} + (Y - \eta)\frac{\partial\phi}{\partial\eta} + (Z - \zeta)\frac{\partial\phi}{\partial\zeta} = 0,$$

which is equivalent to

$$\left\{(X - \xi)\frac{\partial}{\partial\xi} + (Y - \eta)\frac{\partial}{\partial\eta} + (Z - \zeta)\frac{\partial}{\partial\zeta}\right\}[\phi(\xi, \eta, \zeta) + \tfrac{4}{27}] = 0.$$

Hence it touches the surface $\phi(x, y, z) + \frac{4}{27} = 0$.

Hence $\phi(x, y, z) + \frac{4}{27} = 0$ is an envelope.

This accounts for the factor $\phi(x, y, z) + \frac{4}{27}$ in the discriminant.

(D). *Examination of the term* $D^2 f_1/Dx\, Da_1$ *for this example.*

The equation $Df/Da = 0$ is, in this case,

$$\phi(x, y, z) + 3[\psi(x, y, z) - a]^2 = 0.$$

Hence a_1 satisfies

$$\phi(x, y, z) + 3[\psi(x, y, z) - a_1]^2 = 0,$$

and $\partial a_1/\partial x$ is determined by

$$\frac{\partial\phi}{\partial x} + 6[\psi(x, y, z) - a_1]\left[\frac{\partial\psi}{\partial x} - \frac{\partial a_1}{\partial x}\right] = 0.$$

Hence, at a point on the locus of binodal lines, *i.e.*, where $\phi(x, y, z) = 0$, $\psi(x, y, z) = a_1$, it follows that $\partial a_1/\partial x$ is infinite.

Calculating $D^2 f/Dx\, Da$, it follows that it is

$$-\frac{\partial\phi}{\partial x} - 6[\psi(x, y, z) - a]\frac{\partial\psi}{\partial x}.$$

Hence, $D^2 f_1/Dx\, Da_1$ is equal to the value of $-\partial\phi/\partial x$ at the point on a locus of binodal lines. Hence it is finite.

(E). *Examination of the values of* $f_2\Big/\frac{D^2 f_1}{Da_1^2}$, $f_2\Big/\left(\frac{D^2 f_1}{Da_1^2}\right)^2$, $f_2\Big/\left(\frac{D^2 f_1}{Da_1^2}\right)^3$.

In this case a_1, a_2 are the roots of

$$\phi(\xi, \eta, \zeta) + 3[\psi(\xi, \eta, \zeta) - a]^2 = 0.$$

Therefore,

$$a_1 = \psi(\xi, \eta, \zeta) - \sqrt{\{-\tfrac{1}{3}\phi(\xi, \eta, \zeta)\}},$$

$$a_2 = \psi(\xi, \eta, \zeta) + \sqrt{\{-\tfrac{1}{3}\phi(\xi, \eta, \zeta)\}}.$$

Hence

$$f(x, y, z, a_2) = [\phi(\xi, \eta, \zeta)]^2 + \phi(\xi, \eta, \zeta)[\psi(\xi, \eta, \zeta) - a_2] + [\psi(\xi, \eta, \zeta) - a_2]^3$$

$$= [\phi(\xi, \eta, \zeta)]^2 - \tfrac{2}{3}\phi(\xi, \eta, \zeta)\sqrt{\{-\tfrac{1}{3}\phi(\xi, \eta, \zeta)\}}$$

$$\frac{D^2 f(x, y, z, a_1)}{Da_1^2} = 6[\psi(\xi, \eta, \zeta) - a_1] = 6\sqrt{\{-\tfrac{1}{3}\phi(\xi, \eta, \zeta)\}}.$$

Hence

$$f_2\Big/\frac{D^2 f_1}{Da_1^2} = \frac{\sqrt{-3}}{6}\Big[\phi(\xi, \eta, \zeta)\Big]^{3/2} - \tfrac{1}{9}\phi(\xi, \eta, \zeta)$$

$$f_2\Big/\left(\frac{D^2 f_1}{Da_1^2}\right)^2 = -\tfrac{1}{12}\phi(\xi, \eta, \zeta) + \tfrac{1}{18}\sqrt{\{-\tfrac{1}{3}\phi(\xi, \eta, \zeta)\}}.$$

$$f_2\Big/\left(\frac{D^2 f_1}{Da_1^2}\right)^3 = -\frac{\sqrt{-3}}{72}[\phi(\xi, \eta, \zeta)]^{1/2} + \tfrac{1}{108}.$$

Hence, at points on the locus of binodal lines, *i.e.*, when $\phi(\xi, \eta, \zeta) = 0$;

$$f_2\Big/\frac{D^2 f_1}{Da_1^2} = 0,\ f_2\Big/\left(\frac{D^2 f_1}{Da_1^2}\right)^2 = 0,\ J_2\Big/\left(\frac{D^2 f_1}{Da_1^2}\right)^3 = \tfrac{1}{108}.$$

Art. 9.—*Consideration of Loci of Unodal Lines which are also Envelopes.*

(A.) It will be shown that this is the case reserved in Art. 1, viz., where $D^2 f_1/Da_1^2 = 0$.

For if $D^2 f_1/Da_1^2 = 0$, *i.e.*, $[\alpha, \alpha] = 0$, be substituted in the ratios (38), it follows that

$$[\alpha, \xi] = 0,\ [\alpha, \eta] = 0,\ [\alpha, \zeta] = 0.$$

Substituting these in (24)–(26), it follows that

$$[\xi, \xi](\delta\xi) + [\xi, \eta](\delta\eta) + [\xi, \zeta](\delta\zeta) = 0 \quad . \quad . \quad . \quad . \quad . \quad (72),$$

$$[\eta, \xi](\delta\xi) + [\eta, \eta](\delta\eta) + [\eta, \zeta](\delta\zeta) = 0 \quad . \quad . \quad . \quad . \quad . \quad (73),$$

$$[\zeta, \xi](\delta\xi) + [\zeta, \eta](\delta\eta) + [\zeta, \zeta](\delta\zeta) = 0 \quad . \quad . \quad . \quad . \quad . \quad (74).$$

Now (72)–(74) are equivalent to one equation only by (38). Hence the tangent plane to the locus of unodal lines is

$$[\xi, \xi](X - \xi) + [\xi, \eta](Y - \eta) + [\xi, \zeta](Z - \zeta) = 0.$$

Now the tangent cone at ξ, η, ζ is given by (53).

The left-hand side of its equation is by (38) a perfect square.

Hence the uniplane is

$$[\xi, \xi](X - \xi) + [\xi, \eta](Y - \eta) + [\xi, \zeta](Z - \zeta) = 0 \quad . \quad . \quad . \quad (75).$$

Hence the uniplane is the same as the tangent plane to the locus of unodal lines. Hence the locus of unodal lines is also an envelope.

(B.) The converse proposition, viz., that if the locus of unodal lines be also an envelope, then $[\alpha, \alpha] = 0$, will now be proved.

If $\xi + \delta\xi,\ \eta + \delta\eta,\ \zeta + \delta\zeta$ be a point on the locus of unodal lines near to ξ, η, ζ, then the equations (24)–(27) hold.

If the locus of unodal lines be also an envelope, then the equation of the uniplane (75) is satisfied by the values $X = \xi + \delta\xi$, $Y = \eta + \delta\eta$, $Z = \zeta + \delta\zeta$.

Therefore

$$[\xi, \xi](\delta\xi) + [\xi, \eta](\delta\eta) + [\xi, \zeta](\delta\zeta) = 0.$$

Comparing this with (24), it follows that

$$[\alpha, \xi] = 0$$

Hence, by (38),

$$[\alpha, \alpha] = 0.$$

(C.) In this case, the values of $\partial a_1/\partial x$, $\partial a_1/\partial y$ given by (47), (48), are indeterminate. For, because $[\alpha, \alpha] = 0$, it follows by (38) that $[\alpha, \xi] = 0$, $[\alpha, \eta] = 0$, $[\alpha, \zeta] = 0$. Hence, to determine $\partial a_1/\partial x$, it is necessary to differentiate (44) with regard to x. Therefore

$$[\xi, \xi, \alpha] + 2[\xi, \alpha, \alpha]\frac{\partial a_1}{\partial x} + [\alpha, \alpha, \alpha]\left(\frac{\partial a_1}{\partial x}\right)^2 + [\alpha, \alpha]\frac{\partial^2 a_1}{\partial x^2} = 0 \quad . \quad . \quad (76).$$

Hence, because $[\alpha, \alpha] = 0$, and assuming that $\partial^2 a_1/\partial x^2$ is finite, $\partial a_1/\partial x$ satisfies the equation

$$[\xi, \xi, \alpha] + 2[\xi, \alpha, \alpha]\frac{\partial a_1}{\partial x} + [\alpha, \alpha, \alpha]\left(\frac{\partial a_1}{\partial x}\right)^2 = 0 \; . \quad . \quad . \qquad (77).$$

Similarly $\partial a_2/\partial x$ satisfies (77).

Hence, when $x = \xi$, $y = \eta$, $z = \zeta$, $a_1 = a_2 = \alpha$, it follows that $\partial a_1/\partial x$, $\partial a_2/\partial x$ are roots of the same quadratic.

They are finite provided

$$D^3 f_1/Da_1^3 \neq 0 \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78).$$

The case excluded is that in which $Df(x, y, z, a)/Da = 0$ is satisfied by three equal values of a, when $x = \xi$, $y = \eta$, $z = \zeta$. This case might be investigated in a similar manner to the case in which the above equation is satisfied by only two equal values of a, when $x = \xi$, $y = \eta$, $z = \zeta$.

D. *Examination of the Differential Coefficients of* Δ.

In this case Δ and its differential coefficients are given by equations (51), (52). (70), and (71). From these it can be seen, without solving the quadratic (77) for $\partial a_1/\partial x$, $\partial a_2/\partial x$, that Δ, $\partial\Delta/\partial x$, $\partial^2\Delta/\partial x^2$, $\partial^3\Delta/\partial x^3$ all vanish, when $x = \xi$, $y = \eta$, $z = \zeta$.

In like manner it can be shown that all the third differential coefficients of Δ vanish; and, therefore, if $U = 0$ be the equation of the locus of unodal lines which is also an envelope, Δ contains U^4 as a factor.

Example 6.—*Locus of Unodal Lines which is also an Envelope.*

Let the surfaces be

$$[\phi(x, y, z)]^2 - [\psi(x, y, z) - a]^3 = 0.$$

(A.) *The Discriminant.*

The discriminant is found by eliminating a between the above equation and

$$3\,[\psi(x, y, z) - a]^2 = 0.$$

Hence it is $[\phi(x, y, z)]^4$.

Hence the locus of ultimate intersections is

$$[\phi(x, y, z)]^4 = 0.$$

(B.) *The locus of Unodal Lines which is also an Envelope is* $\phi(x, y, z) = 0$.

For, let ξ, η, ζ be any point on the surface $\phi(x, y, z) = 0$. Take $a = \psi(\xi, \eta, \zeta)$, and consider the single surface

$$[\phi(x, y, z)]^2 - [\psi(x, y, z) - \psi(\xi, \eta, \zeta)]^3 = 0.$$

Put

$$x = \xi + \mathrm{X},\quad y = \eta + \mathrm{Y},\quad z = \zeta + \mathrm{Z}.$$

Then the lowest terms in X, Y, Z are

$$\left(\mathrm{X}\frac{\partial\phi}{\partial\xi} + \mathrm{Y}\frac{\partial\phi}{\partial\eta} + \mathrm{Z}\frac{\partial\phi}{\partial\zeta}\right)^2.$$

Hence ξ, η, ζ is a unode on the surface

$$[\phi(x, y, z)]^2 - [\psi(x, y, z) - \psi(\xi, \eta, \zeta)]^3 = 0.$$

Hence the locus of unodal lines of the surfaces under consideration is $\phi(x, y, z) = 0$.

Moreover, the uniplane

$$\mathrm{X}\frac{\partial\phi}{\partial\xi} + \mathrm{Y}\frac{\partial\phi}{\partial\eta} + \mathrm{Z}\frac{\partial\phi}{\partial\zeta} = 0$$

is also a tangent plane to the locus of unodal lines.

Hence the locus of unodal lines is also an envelope.

Hence the factor $[\phi(x, y, z)]^4$ in the discriminant is accounted for.

Section III. (Arts. 10–11).—Supplementary Remarks.

Art. 10.—*Further remark on the case in which* $\mathrm{D}^2 f_1/\mathrm{D}a_1^2 = 0$.

This condition indicates in general that the equation $\mathrm{D}f_1/\mathrm{D}a_1 = 0$ has two equal roots, but if f_1 be of the second degree in a_1, $\mathrm{D}f_1/\mathrm{D}a_1$ is of the first degree in a_1, and hence it has either one root in a_1, or is satisfied by an infinite number of values of a_1.

It is desirable to notice the latter case, because it corresponds to an important case treated in Part II., Section IV., of this paper.

Let $f = Ua^2 + 2Va + W = 0$, where U, V, W are rational integral functions of x, y, z.

Then the conditions $f = 0$, $Df/Da = 0$, $D^2f/Da^2 = 0$ are equivalent to

$$\begin{aligned} Ua^2 + 2Va + W &= 0, \\ Ua + V &= 0, \\ U &= 0, \end{aligned}$$

at all points of the locus of ultimate intersections.

Hence $U = 0$, $V = 0$, $W = 0$ at all such points.

Hence unless U, V, W have a common factor, which could in that case be removed from the equation $f = 0$, the locus of ultimate intersections is not a surface, and hence its equation cannot be obtained by equating a factor of the discriminant to zero. Hence this case need not be further considered.

Art. 11.—*If the surface $f(x, y, z) = 0$ have upon it a curve at every point of which there is a conic node, then the tangent cones at the conic nodes must break up into two planes.**

Let ξ, η, ζ; $\xi + \delta\xi, \eta + \delta\eta, \zeta + \delta\zeta$; be neighbouring points on the curve.

Then since there is a conic node at ξ, η, ζ;

$$f(\xi, \eta, \zeta) = 0,\ Df/D\xi = 0,\ Df/D\eta = 0,\ Df/D\zeta = 0.$$

And since there is a conic node at $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$, four other equations hold, which by means of the above give

$$\begin{aligned} &[\xi, \xi](\delta\xi)^2 + [\eta, \eta](\delta\eta)^2 + [\zeta, \zeta](\delta\zeta)^2 \\ &+ 2[\eta, \zeta](\delta\eta)(\delta\zeta) + 2[\zeta, \xi](\delta\zeta)(\delta\xi) + 2[\xi, \eta](\delta\xi)(\delta\eta) + \ldots = 0, \\ &[\xi, \xi](\delta\xi) + [\xi, \eta](\delta\eta) + [\xi, \zeta](\delta\zeta) + \ldots = 0, \\ &[\eta, \xi](\delta\xi) + [\eta, \eta](\delta\eta) + [\eta, \zeta](\delta\zeta) + \ldots = 0, \\ &[\zeta, \xi](\delta\xi) + [\zeta, \eta](\delta\eta) + [\zeta, \zeta](\delta\zeta) + \ldots = 0. \end{aligned}$$

Retaining only the principal terms in the last three equations, it follows that

* The geometry of a surface of continuous curvature shows at once that there cannot be a curve of conical points on a surface.

$$[\xi, \xi](\delta\xi) + [\xi, \eta](\delta\eta) + [\xi, \zeta](\delta\zeta) = 0,$$
$$[\eta, \xi](\delta\xi) + [\eta, \eta](\delta\eta) + [\eta, \zeta](\delta\zeta) = 0,$$
$$[\zeta, \xi](\delta\xi) + [\zeta, \eta](\delta\eta) + [\zeta, \zeta](\delta\zeta) = 0.$$

These equations must be consistent, and therefore

$$\frac{D[[\xi], [\eta], [\zeta]]}{D[\xi, \eta, \zeta]} = 0.$$

This is the condition that the tangent cone at ξ, η, ζ, viz. :—

$$[\xi, \xi](X - \xi)^2 + [\eta, \eta](Y - \eta)^2 + [\zeta, \zeta](Z - \zeta)^2$$
$$+ 2[\eta, \zeta](Y - \eta)(Z - \zeta) + 2[\zeta, \xi](Z - \zeta)(X - \xi) + 2[\xi, \eta](X - \xi)(Y - \eta) = 0,$$

may break up into two planes.

PART II.—THE EQUATION OF THE SYSTEM OF SURFACES IS A RATIONAL INTEGRAL FUNCTION OF THE COORDINATES AND TWO ARBITRARY PARAMETERS.

SECTION I. (Art. 1).—PRELIMINARY THEOREMS.

Art. 1. (A.) *If ξ, η, ζ are the coordinates of any point on the locus $\phi(x, y, z) = 0$ (where ϕ is a rational integral indecomposable function of x, y, z), and if the substitutions $x = \xi$, $y = \eta$, $z = \zeta$ make $\psi(x, y, z)$ and all its partial differential coefficients with regard to x, y, z up to the n^{th} order vanish, and if they also make any one of the partial differential coefficients of the $(n + 1)^{th}$ order vanish, they will also make all the partial differential coefficients of the $(n + 1)^{th}$ order vanish (ψ being a rational integral function of x, y, z, but not in general indecomposable).*

Suppose that when $x = \xi$, $y = \eta$, $z = \zeta$,

$$\frac{\partial^{n+1}\psi}{\partial x^{r+1}\,\partial y^s\,\partial z^{n-r-s}} = 0,$$

where ∂ denotes partial differentiation when x, y, z are independent variables.

To prove that the same substitutions make

$$\frac{\partial^{n+1}\psi}{\partial x^{r}\,\partial y^{s+1}\,\partial z^{n-r-s}} = 0,$$

and

$$\frac{\partial^{n+1}\psi}{\partial x^{r}\,\partial y^{s}\,\partial z^{n-r-s+1}} = 0.$$

It is given that all the values of x, y, z which make $\phi(x, y, z) = 0$, also make

$$\frac{\partial^{n+1}\psi}{\partial x^{r+1}\partial y^s\partial z^{n-r-s}} = 0.$$

Now let $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$ be a point near to ξ, η, ζ on $\phi(x, y, z) = 0$.
Therefore $\phi(\xi, \eta, \zeta) = 0$.

$$\phi(\xi + \delta\xi, \eta + \delta\eta, \zeta + \delta\zeta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

$$\frac{\partial^{n+1}\psi}{\partial\xi^{r+1}\partial\eta^s\partial\zeta^{n-r-s}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

And since ξ, η, ζ make

$$\frac{\partial^{n}\psi}{\partial x^{r}\partial y^s\partial z^{n-r-s}} = 0,$$

for this is a differential coefficient of the n^{th} order, $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$ must also do the same.

Hence

$$(\delta\xi)\frac{\partial^{n+1}\psi}{\partial\xi^{r+1}\partial\eta^s\partial\zeta^{n-r-s}} + (\delta\eta)\frac{\partial^{n+1}\psi}{\partial\xi^{r}\partial\eta^{s+1}\partial\zeta^{n-r-s}} + (\delta\zeta)\frac{\partial^{n+1}\psi}{\partial\xi^{r}\partial\eta^s\partial\zeta^{n-r-s+1}} = 0.$$

Also from (1)

$$(\delta\xi)\frac{\partial\phi}{\partial\xi} + (\delta\eta)\frac{\partial\phi}{\partial\eta} + (\delta\zeta)\frac{\partial\phi}{\partial\zeta} = 0.$$

Since this is the only relation between $\delta\xi$, $\delta\eta$, $\delta\zeta$, it follows that

$$\frac{\left(\dfrac{\partial^{n+1}\psi}{\partial\xi^{r+1}\partial\eta^s\partial\zeta^{n-r-s}}\right)}{\left(\dfrac{\partial\phi}{\partial\xi}\right)} = \frac{\left(\dfrac{\partial^{n+1}\psi}{\partial\xi^{r}\partial\eta^{s+1}\partial\zeta^{n-r-s}}\right)}{\left(\dfrac{\partial\phi}{\partial\eta}\right)} = \frac{\left(\dfrac{\partial^{n+1}\psi}{\partial\xi^{r}\partial\eta^s\partial\zeta^{n-r-s+1}}\right)}{\left(\dfrac{\partial\phi}{\partial\zeta}\right)}.$$

Hence, by means of (2),

$$\frac{\partial^{n+1}\psi}{\partial\xi^{r}\partial\eta^{s+1}\partial\zeta^{n-r-s}} = 0, \qquad \frac{\partial^{n+1}\psi}{\partial\xi^{r}\partial\eta^s\partial\zeta^{n-r-s+1}} = 0.$$

In this way it is possible to pass from any one partial differential coefficient of order $(n + 1)$ by successive steps to any other of order $(n + 1)$; at each step always diminishing by one the number of differentiations with regard to one variable, and increasing by one the number of differentiations with regard to another variable.

Hence all the differential coefficients of the $(n + 1)^{\text{th}}$ order vanish when $x = \xi$, $y = \eta$, $z = \zeta$.

(B.) (i.) *If* ξ, η, ζ, *are the co-ordinates of any point on the locus* $\phi(x, y, z) = 0$ (*where* ϕ *is a rational integral function of* x, y, z *which contains no repeated factors*), *and if the substitutions* $x = \xi$, $y = \eta$, $z = \zeta$ *make* $\psi(x, y, z) = 0$ (*where* ψ *is a rational integral function of* x, y, z), *then* ψ *contains the first power of* ϕ *as a factor.*

(ii.) *If* $x = \xi$, $y = \eta$, $z = \zeta$ *make* $\psi = 0$, $\dfrac{\partial\psi}{\partial x} = 0$, *then* ψ *contains the second power of* ϕ *as a factor.*

(iii.) *If* $x = \xi$, $y = \eta$, $z = \zeta$ *make* $\psi = 0$, $\dfrac{\partial\psi}{\partial x} = 0, \ldots \dfrac{\partial^{m-1}\psi}{\partial x^{m-1}} = 0$, *then* ψ *contains* ϕ^m *as a factor.*

To prove (i.) suppose first that ϕ is indecomposable. It is obvious that ψ cannot be of lower dimensions than ϕ in any one of the variables; if it were, then all the values of x, y, z which make $\phi = 0$ would not make $\psi = 0$.

It may happen that ϕ does not contain all the variables x, y, z. But it must contain one of them; suppose it contains x.

If ϕ be not a factor of ψ, proceed as in the process for finding the common factor of highest dimensions in x of ϕ and ψ; and if, at any step of the process, fractional quotients in which the denominators are functions of y, z are obtained, let the denominators be removed in the usual way by multiplication throughout by a factor.

Then either the process will terminate, or there will at last be a remainder, which is a function of y, z only, not x.

In the first case ϕ and ψ will have a common factor, and ϕ will be decomposable, which is contrary to the hypothesis.

In the second case a relation of the form

$$A\psi = B\phi + C$$

exists, where A, B, C are rational integral functions of y, z only. In this case, since all the values of x, y, z which make $\phi = 0$, also make $\psi = 0$, therefore they make $C = 0$. But C is a function of y, z only, not x. Now, the values under consideration are values of x, y, z. This is impossible. Hence this alternative does not hold.

Hence ϕ must be a factor of ψ.

If ϕ be decomposable, its indecomposable factors may be taken separately, and shown as above to be factors of ψ.

As it is further supposed that ϕ contains no repeated factors, it follows that ψ contains ϕ as a factor.

To prove (ii.).

By the same argument as in (i.) it follows that ψ contains ϕ as a factor. Let $\psi = R\phi$.

Therefore,

$$\frac{\partial\psi}{\partial x} = \phi\frac{\partial R}{\partial x} + R\frac{\partial\phi}{\partial x}.$$

Now the substitutions $x = \xi$, $y = \eta$, $z = \zeta$ make $\phi = 0$, $\psi = 0$, $\partial\psi/\partial x = 0$. Therefore they make

$$R\frac{\partial\phi}{\partial x} = 0.$$

Now all the values of x, y, z which make $\phi = 0$ cannot make $\partial\phi/\partial x = 0$. Hence $x = \xi$, $y = \eta$, $z = \zeta$ must make R vanish.

Therefore R is divisible by ϕ without remainder.

Therefore ψ is divisible by ϕ^2 without remainder.

To prove (iii.) proceed by induction.

Suppose that the theorem is true for a given value of m, viz., that if $x = \xi$, $y = \eta$, $z = \zeta$ make $\psi = 0$, $\partial\psi/\partial x = 0$, . . . $\partial^{m-1}\psi/\partial x^{m-1} = 0$, then ψ contains ϕ^m as a factor.

Let it now be given that $x = \xi$, $y = \eta$, $z = \zeta$ also make $\partial^m\psi/\partial x^m = 0$.

Then by the assumption

$$\psi = \rho.\phi^m,$$

where ρ is some rational integral function of x, y, z.

Therefore

$$\frac{\partial^m\psi}{\partial x^m} = \rho\frac{\partial^m\phi^m}{\partial x^m} + m\frac{\partial\rho}{\partial x}\frac{\partial^{m-1}\phi^m}{\partial x^{m-1}} + \dots$$
$$= \rho\frac{\partial^m\phi^m}{\partial x^m} + \phi\chi,$$

where χ is some rational integral function of x, y, z.

Now

$$\frac{\partial^m\phi^m}{\partial x^m} = m\,!\left(\frac{\partial\phi}{\partial x}\right)^m + \phi.\sigma,$$

where σ is some rational integral function of x, y, z.

Therefore

$$\frac{\partial^m\psi}{\partial x^m} = m\,!\,\rho\left(\frac{\partial\phi}{\partial x}\right)^m + \phi(\rho\sigma + \chi).$$

Hence the substitutions $x = \xi$, $y = \eta$, $z = \zeta$ make

$$\rho\left(\frac{\partial\phi}{\partial x}\right)^m = 0,$$

but they do not make $\partial\phi/\partial x = 0$.

Therefore they make $\rho = 0$.

Therefore ρ contains ϕ as a factor.

Therefore ψ contains ϕ^{m+1} as a factor.

Hence, if the theorem is true for a special value of m, it is true for the next value.

But it has been proved true when $m = 2$, hence it is true in general.

(C.) *If u, v be determined as functions of other quantities by the equations*

$$\phi(u, v) = 0, \quad \psi(u, v) = 0,$$

where ϕ and ψ are rational integral functions of u, v and the other quantities; then, if two systems of common values of u, v become equal, they will also satisfy the equation

$$\frac{D[\phi, \psi]}{D[u, v]} = 0.$$

Conversely, if values of u, v can be found to satisfy at the same time the three equations

$$\phi(u, v) = 0, \quad \psi(u, v) = 0, \quad \frac{D[\phi, \psi]}{D[u, v]} = 0,$$

then these values count twice over among the common solutions of the equations

$$\phi(u, v) = 0, \quad \psi(u, v) = 0,$$

except in the case where ϕ and ψ are of the first degree in u and v; and then the two equations have an infinite number of solutions in common.

To prove this, let u, v represent the coordinates of a point in a plane. Then $\phi(u, v) = 0$, $\psi(u, v) = 0$ are the equations of two algebraic curves.

The values of u, v which satisfy at the same time both equations are the coordinates of the points of intersection of the two curves.

Let $u = \alpha$, $v = \beta$ be the coordinates of one point of intersection. The tangents to the curves at α, β are

$$(U - \alpha)\frac{\partial\phi(\alpha, \beta)}{\partial\alpha} + (V - \beta)\frac{\partial\phi(\alpha, \beta)}{\partial\beta} = 0,$$

$$(U - \alpha)\frac{\partial\psi(\alpha, \beta)}{\partial\alpha} + (V - \beta)\frac{\partial\psi(\alpha, \beta)}{\partial\beta} = 0,$$

where U, V are current coordinates.

The two tangents will coincide, *i.e.*, the curves have two coincident points of intersection, if

$$\frac{\partial\phi(\alpha, \beta)}{\partial\alpha}\frac{\partial\psi(\alpha, \beta)}{\partial\beta} - \frac{\partial\phi(\alpha, \beta)}{\partial\beta}\frac{\partial\psi(\alpha, \beta)}{\partial\alpha} = 0,$$

i.e., if $u = \alpha$, $v = \beta$ satisfy

$$\frac{\mathrm{D}\,[\phi, \psi]}{\mathrm{D}\,[u, v]} = 0.$$

In the case where ϕ, ψ are each of the first degree in u, v, then the theorem requires to be specially interpreted, the interpretation corresponding to the fact that if two straight lines have two points in common, they have an infinite number of points in common. Hence, in this case, the two equations have an infinite number of solutions in common. (This particular case is of great importance in Section IV. of this investigation.)

(D.) *To determine the conditions that the equations* $\phi(u, v) = 0$, $\psi(u, v) = 0$, *may be satisfied by three coinciding systems of common values.*

In this case, considering as in (C) that u, v represent the coordinates of a point, the curves $\phi(u, v) = 0$, $\psi(u, v) = 0$ must have contact of the second order.

Now

$$\frac{\partial\phi}{\partial u} + \frac{\partial\phi}{\partial v}\frac{dv}{du} = 0$$

$$\frac{\partial^2\phi}{\partial u^2} + 2\frac{\partial^2\phi}{\partial u\,\partial v}\frac{dv}{du} + \frac{\partial^2\phi}{\partial v^2}\left(\frac{dv}{du}\right)^2 + \frac{\partial\phi}{\partial v}\frac{d^2v}{du^2} = 0.$$

Therefore

$$\frac{d^2v}{du^2} = -\left[\frac{\partial^2\phi}{\partial u^2}\left(\frac{\partial\phi}{\partial v}\right)^2 - 2\frac{\partial^2\phi}{\partial u\,\partial v}\frac{\partial\phi}{\partial u}\frac{\partial\phi}{\partial v} + \frac{\partial^2\phi}{\partial v^2}\left(\frac{\partial\phi}{\partial u}\right)^2\right]\left(\frac{\partial\phi}{\partial v}\right)^{-3}.$$

Hence, equating the values of d^2v/du^2 for the two curves, there is obtained the further condition

$$\left[\frac{\partial^2\phi}{\partial u^2}\left(\frac{\partial\phi}{\partial v}\right)^2 - 2\frac{\partial^2\phi}{\partial u\,\partial v}\frac{\partial\phi}{\partial u}\frac{\partial\phi}{\partial v} + \frac{\partial^2\phi}{\partial v^2}\left(\frac{\partial\phi}{\partial u}\right)^2\right]\left(\frac{\partial\phi}{\partial v}\right)^{-3}$$
$$= \left[\frac{\partial^2\psi}{\partial u^2}\left(\frac{\partial\psi}{\partial v}\right)^2 - 2\frac{\partial^2\psi}{\partial u\,\partial v}\frac{\partial\psi}{\partial u}\frac{\partial\psi}{\partial v} + \frac{\partial^2\psi}{\partial v^2}\left(\frac{\partial\psi}{\partial u}\right)^2\right]\left(\frac{\partial\psi}{\partial v}\right)^{-3}.$$

SECTION II. (Arts. 2–12).—THE FACTORS OF THE DISCRIMINANT, WHICH IN GENERAL CORRESPOND TO ENVELOPE AND SINGULAR POINT LOCI.

Art. 2.—*The Fundamental Equations.*

Let the equation of the system of surfaces be

$$f(x, y, z, a, b) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where a, b are independent arbitrary parameters, and f is a rational integral indecomposable function of x, y, z, a, b.

The locus of ultimate intersections is obtained by eliminating a and b between (3) and

$$\frac{\mathrm{D}f(x, y, z, a, b)}{\mathrm{D}a} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

$$\frac{\mathrm{D}f(x, y, z, a, b)}{\mathrm{D}b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

where D denotes partial differentiation when x, y, z, a, b are treated as independent variables.

Let the result of the elimination be $\Delta = 0$, then Δ is called the discriminant.

If x, y, z are chosen so as to make any factor of the discriminant vanish, it indicates *in general* that it is possible to satisfy equations (3), (4), (5) by the same values of a, b. Hence x, y, z can be expressed as functions of a, b.

In this case $x = \phi(a, b)$, $y = \psi(a, b)$, $z = \chi(a, b)$.

Eliminating a and b a surface locus is obtained.

This is the general case. The exceptional cases are noticed in Section VI., Art. 30.

Art. 3.—*The Loci of Singular Points of the System of Surfaces.*

The equation of the locus of singular points on the surfaces (3) can be obtained by eliminating a and b, between (3), and

$$\frac{\mathrm{D}f(x, y, z, a, b)}{\mathrm{D}x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

$$\frac{\mathrm{D}f(x, y, z, a, b)}{\mathrm{D}y} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

$$\frac{\mathrm{D}f(x, y, z, a, b)}{\mathrm{D}z} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

The singular points are in general conic nodes.

The locus of conic nodes is therefore a curve, whose equations are given by eliminating a, b between (3), (6), (7), (8).

It follows, also, by eliminating x, y, z between the same equations, that there is a definite relation between a, b.

If ξ, η, ζ be the coordinates of the conic node on the surface

$$f(x, y, z, \alpha, \beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

and $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$ the coordinates of the conic node on the surface

$$f(x, y, z, \quad \alpha + \delta\alpha, \quad \beta + \delta\beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

then the following equations all hold at the same time,

$$f(\xi, \eta, \zeta, \alpha, \beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

$$\frac{Df(\xi, \eta, \zeta, \alpha, \beta)}{D\xi} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

$$\frac{Df(\xi, \eta, \zeta, \alpha, \beta)}{D\eta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

$$\frac{Df(\xi, \eta, \zeta, \alpha, \beta)}{D\zeta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

and four other equations, which, by means of the above, become

$$\frac{Df}{D\alpha}(\delta\alpha) + \frac{Df}{D\beta}(\delta\beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

$$[\xi, \xi](\delta\xi) + [\xi, \eta](\delta\eta) + [\xi, \zeta](\delta\zeta) + [\xi, \alpha](\delta\alpha) + [\xi, \beta](\delta\beta) = 0 \quad . \quad (16),$$

$$[\eta, \xi](\delta\xi) + [\eta, \eta](\delta\eta) + [\eta, \zeta](\delta\zeta) + [\eta, \alpha](\delta\alpha) + [\eta, \beta](\delta\beta) = 0 \quad . \quad (17),$$

$$[\zeta, \xi](\delta\xi) + [\zeta, \eta](\delta\eta) + [\zeta, \zeta](\delta\zeta) + [\zeta, \alpha](\delta\alpha) + [\zeta, \beta](\delta\beta) = 0 \quad . \quad (18).$$

If $\delta\alpha$, $\delta\beta$ be eliminated from (15)–(18), the ratios $\delta\xi : \delta\eta : \delta\zeta$ are determined. These ratios determine the tangent line to the curve locus of conic nodes.

If β be determined as a function of α, so that (11)–(14) can be satisfied by the same values of ξ, η, ζ, then the equations (11) and (15) show that the locus of conic nodes is a curve lying on one of the *general integrals* of the partial differential equation of the surfaces (3).

Example 1. *Curve Locus of Conic Nodes.*

Let the surfaces be

$$x^2 + e(y - \tfrac{1}{2}a)^2 + (z - \tfrac{1}{2}b)^2 - cx + a + b = 0 \quad . \quad . \quad . \quad . \quad (19),$$

where e, c are fixed constants; a, b the arbitrary parameters.

(A.) *The locus of conic nodes is the straight line.*

$$x = \tfrac{1}{2} c, \quad y + z = \tfrac{1}{8} c^2.$$

To find this locus it is necessary to eliminate a, b between (19) and

$$\frac{Df}{Dx} = 2x - c = 0 \quad \ldots \ldots \ldots \quad (20),$$

$$\frac{Df}{Dy} = 2e\,(y - \tfrac{1}{2} a) = 0 \quad \ldots \ldots \ldots \quad (21),$$

$$\frac{Df}{Dz} = 2\,(z - \tfrac{1}{2} b) = 0 \quad \ldots \ldots \ldots \quad (22).$$

Therefore

$$x = \tfrac{1}{2} c, \quad y = \tfrac{1}{2} a, \quad z = \tfrac{1}{2} b, \quad y + z = \tfrac{1}{8} c^2.$$

Hence the locus of conic nodes is the straight line,

$$x = \tfrac{1}{2} c, \quad y + z = \tfrac{1}{8} c^2 \quad \ldots \ldots \ldots \quad (23).$$

(B.) *The locus of conic nodes lies on the general integral of the partial differential equation of the surfaces* (19) *obtained by putting* $b = \tfrac{1}{4} c^2 - a$.

To determine this general integral take the values of x, y, z from (20)–(22), and substitute in (19). This gives $a + b = \tfrac{1}{4} c^2$.

Hence the general integral is obtained by eliminating a from

$$x^2 + e\,(y - \tfrac{1}{2} a)^2 + (z + \tfrac{1}{2} a - \tfrac{1}{8} c^2)^2 - cx + \tfrac{1}{4} c^2 = 0,$$

and

$$- e\,(y - \tfrac{1}{2} a) + (z + \tfrac{1}{2} a - \tfrac{1}{8} c^2) = 0.$$

Hence it is

$$e\,(y + z - \tfrac{1}{8} c^2)^2 + (1 + e)\,(x - \tfrac{1}{2} c)^2 = 0.$$

It contains the locus of conic nodes, whose equations are given in (23).

(C.) *The locus of conic nodes does not lie on the locus of ultimate intersections.*

For the equation of the locus of ultimate intersections is obtained by eliminating a, b between

$$f = x^2 + e\,(y - \tfrac{1}{2} a)^2 + (z - \tfrac{1}{2} b)^2 - cx + a + b = 0,$$

$$\frac{Df}{Da} = - e\,(y - \tfrac{1}{2} a) + 1 = 0,$$

$$\frac{Df}{Db} = - (z - \tfrac{1}{2} b) + 1 = 0.$$

It is therefore

$$x^2 - cx + 2y + 2z - 1 - \frac{1}{e} = 0.$$

This does not contain the locus of conic nodes.

It is an envelope touching f at the two points

$$x = \frac{c}{2} \pm \left(\frac{c^2}{4} - 1 - \frac{1}{e} - a - b\right)^{\frac{1}{2}}.$$

$$y = \tfrac{1}{2}a + \frac{1}{e}.$$

$$z = \tfrac{1}{2}b + 1.$$

Art. 4.—*Investigation of the conditions which are satisfied at any point on the Locus of Conic Nodes.*

In the preceding article it was shown that the surfaces (3) have in general a curve locus of conic nodes.

If, however, *every one* of the surfaces (3) has a conic node, then equations (3), (6), (7), (8) are equivalent to three independent equations only, and the locus of conic nodes is a surface, whose equation is obtained by eliminating a and b between any three of the four equations (3), (6), (7), (8).

It will be proved that such a surface locus of conic nodes is a part at least of the locus of ultimate intersections.

With the notation of the last article, equations (11)–(18) hold; but now there is no relation between α and β.

There is a conic node on the surface

$$f(x, y, z, \alpha + \delta\alpha, \beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

Hence (15) must hold when $\delta\beta = 0$.

Hence

$$\mathrm{D}f/\mathrm{D}\alpha = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

Similarly

$$\mathrm{D}f/\mathrm{D}\beta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

Since (11), (25), and (26) hold at all points of the conic node locus, it follows that the conic node locus is a part, at least, of the locus of ultimate intersections.

The position of the tangent plane to the conic node locus may be obtained from (16)–(18) by eliminating $\delta\alpha$, $\delta\beta$; and then using the relations

$$\frac{\delta\xi}{X-\xi}=\frac{\delta\eta}{Y-\eta}=\frac{\delta\zeta}{Z-\zeta} \quad . \quad . \quad . \quad . \quad (27),$$

where X, Y, Z are current coordinates.

Since (25) and (26) are satisfied at all points of the conic node locus, they are satisfied when $\xi, \eta, \zeta, \alpha, \beta$ are replaced by $\xi+\delta\xi, \eta+\delta\eta, \zeta+\delta\zeta, \alpha+\delta\alpha, \beta+\delta\beta$ respectively.

Hence

$$[\alpha, \xi](\delta\xi)+[\alpha, \eta](\delta\eta)+[\alpha, \zeta](\delta\zeta)+[\alpha, \alpha](\delta\alpha)+[\alpha, \beta](\delta\beta)=0 \quad . \quad (28).$$

$$[\beta, \xi](\delta\xi)+[\beta, \eta](\delta\eta)+[\beta, \zeta](\delta\zeta)+[\beta, \alpha](\delta\alpha)+[\beta, \beta](\delta\beta)=0 \quad . \quad (29).$$

Now (28) and (29) are not independent of (16)-(18).

For if $\delta\alpha$, $\delta\beta$ are definite infinitely small quantities, then (16)-(18) determine the values of $\delta\xi$, $\delta\eta$, $\delta\zeta$ corresponding to the conic node on the surface (10). Substituting these values in (28) and (29), and observing that $\delta\alpha$, $\delta\beta$ being independent may be supposed to vanish separately, the following relations are obtained (using the usual notation for Jacobians):—

$$\frac{D[[\xi],[\eta],[\zeta],[\alpha]]}{D[\xi,\eta,\zeta,\alpha]}=0 \quad . \quad . \quad . \quad . \quad (30),$$

$$\frac{D[[\xi],[\eta],[\zeta],[\beta]]}{D[\xi,\eta,\zeta,\alpha]}=0 \quad . \quad . \quad . \quad . \quad (31),$$

$$\frac{D[[\xi],[\eta],[\zeta],[\beta]]}{D[\xi,\eta,\zeta,\beta]}=0 \quad . \quad . \quad . \quad . \quad (32).$$

Other similar relations exist which may be found by taking any four of the equations (16), (17), (18), (28), (29), putting any one of the five quantities $\delta\xi$, $\delta\eta$, $\delta\zeta$, $\delta\alpha$, $\delta\beta$ equal to zero, and expressing that the equations give consistent values for the four quantities which remain.

Hence any minor of the fourth order of the Jacobian

$$\frac{D[[\xi],[\eta],[\zeta],[\alpha],[\beta]]}{D[\xi,\eta,\zeta,\alpha,\beta]}$$

vanishes.

Art. 5.—*Investigation of the conditions which are satisfied at any point on the Locus of Biplanar Nodes.*

The equation of the tangent cone which, in this case, becomes the equation of the biplanes, at the singular point is

$$[\xi, \xi](X-\xi)^2 + [\eta, \eta](Y-\eta)^2 + [\zeta, \zeta](Z-\zeta)^2$$
$$+2[\eta,\zeta](Y-\eta)(Z-\zeta)+2[\zeta,\xi](Z-\zeta)(X-\xi)+2[\xi,\eta](X-\xi)(Y-\eta)=0 \quad . \quad (33)$$

This breaks up into factors, linear with regard to $X-\xi$, $Y-\eta$, $Z-\zeta$.

Therefore

$$\frac{D[[\xi],[\eta],[\zeta]]}{D[\xi,\eta,\zeta]}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).$$

Now if in (16)–(18), $\delta\beta$ be put equal to zero (which is possible, since there is by hypothesis a biplanar node on the surface (24)), the values of $\delta\xi/\delta\alpha$, $\delta\eta/\delta\alpha$, $\delta\zeta/\delta\alpha$ must be finite.

But the denominators of the values of these expressions vanish by (34). Hence their numerators also vanish.

Therefore

$$\frac{D[[\xi],[\eta],[\alpha]]}{D[\xi,\eta,\zeta]} \equiv \frac{D[[\xi],[\eta],[\zeta]]}{D[\xi,\eta,\alpha]}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35),$$

$$\frac{D[[\xi],[\zeta],[\alpha]]}{D[\xi,\eta,\zeta]} \equiv \frac{D[[\xi],[\eta],[\zeta]]}{D[\xi,\zeta,\alpha]}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36),$$

$$\frac{D[[\eta],[\zeta],[\alpha]]}{D[\xi,\eta,\zeta]} \equiv \frac{D[[\xi],[\eta],[\zeta]]}{D[\eta,\zeta,\alpha]}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37).$$

And similarly it can be shown that the following equations obtained by changing into β in the above also hold good :—

$$\frac{D[[\xi],[\eta],[\beta]]}{D[\xi,\eta,\zeta]} \equiv \frac{D[[\xi],[\eta],[\zeta]]}{D[\xi,\eta,\beta]}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38),$$

$$\frac{D[[\xi],[\zeta],[\beta]]}{D[\xi,\eta,\zeta]} \equiv \frac{D[[\xi],[\eta],[\zeta]]}{D[\xi,\zeta,\beta]}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39),$$

$$\frac{D[[\eta],[\zeta],[\beta]]}{D[\xi,\eta,\zeta]} \equiv \frac{D[[\xi],[\eta],[\zeta]]}{D[\eta,\zeta,\beta]}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40).$$

From these it follows that (16)–(18) are equivalent to two independent equations only in this case.

Now consider equations (16), (17), (28).

The equation (35) makes the determinant formed from the coefficients of $\delta\xi$, $\delta\eta$, $\delta\zeta$ in them vanish. Hence it bears to them the same relation that (34) bears to (16), (17), (18).

Therefore (16), (17), (28) are equivalent to two independent equations only.

In like manner (36) shows that (16), (18), (28) are equivalent to two independent equations only.

Also (37) shows that (17), (18), (28) are equivalent to two independent equations only.

Hence (28) and *any* two of the equations (16), (17), (18) are equivalent to two independent equations only.

Similarly, by means of (38)-(40), it can be shown that (29), and any two of the equations (16), (17), (18) are equivalent to two independent equations only.

Hence the five equations (16), (17), (18), (28), (29) are equivalent to two independent equations only in this case.

Since (16), (28), (29) are not independent, it follows that (amongst other relations)

$$\frac{D[[\xi], [\alpha], [\beta]]}{D[\xi, \alpha, \beta]} = 0 \quad \text{. . . . .} \quad (41),$$

and

$$\frac{D[[\xi], [\alpha], [\beta]]}{D[\eta, \alpha, \beta]} \equiv \frac{D[[\eta], [\alpha], [\beta]]}{D[\xi, \alpha, \beta]} = 0 \quad \text{. . . . . . .} \quad (42).$$

Art. 6.—*Investigation of the conditions which are satisfied at any point on the Locus of Uniplanar Nodes.*

In this case, the left-hand side of the equation (33) becomes a perfect square.

Therefore

$$\begin{aligned} &[\xi, \xi] : [\xi, \eta] : [\xi, \zeta] \\ = &[\eta, \xi] : [\eta, \eta] : [\eta, \zeta] \\ = &[\zeta, \xi] : [\zeta, \eta] : [\zeta, \zeta]. \quad \text{. . . . .} \quad (43). \end{aligned}$$

Now, multiplying (16) by $[\eta, \xi]$, (17) by $[\xi, \xi]$, and subtracting

$$(\delta\alpha)\ \{[\xi, \alpha][\eta, \xi] - [\eta, \alpha][\xi, \xi]\} + (\delta\beta)\ \{[\xi, \beta][\eta, \xi] - [\eta, \beta][\xi, \xi]\} = 0. \quad (44).$$

Now, there is a uniplanar node on the surface (24), hence $\delta\beta$ may be made to vanish.

Therefore

$$[\xi, \alpha][\eta, \xi] - [\eta, \alpha][\xi, \xi] = 0 \quad \text{. . . . .} \quad (45).$$

Similarly

$$[\xi, \beta][\eta, \xi] - [\eta, \beta][\xi, \xi] = 0.$$

Therefore

$$[\xi, \xi] / [\eta, \xi] = [\xi, \alpha] / [\eta, \alpha] = [\xi, \beta] / [\eta, \beta] \quad \text{. . . . .} \quad (46).$$

Now (43) and (46) show that (16) and (17) are equivalent to one independent equation only. Similarly (16) and (18) are equivalent to one independent equation only.

Hence the following ratios hold

$$\begin{aligned}&[\xi, \xi] : [\xi, \eta] : [\xi, \zeta] : [\xi, \alpha] : [\xi, \beta]\\ =\ &[\eta, \xi] : [\eta, \eta] : [\eta, \zeta] : [\eta, \alpha] : [\eta, \beta]\\ =\ &[\zeta, \xi] : [\zeta, \eta] : [\zeta, \zeta] : [\zeta, \alpha] : [\zeta, \beta] \quad . \quad . \quad . \quad . \quad . \quad . \quad (47).\end{aligned}$$

Applying (47) to (16) and (28), it follows that these two are equivalent to one independent equation only.

Similarly (16) and (29) are equivalent to one independent equation only.

Hence the five equations (16), (17, (18), (28), (29), are equivalent to one independent equation only, and, therefore, the following ratios hold

$$\begin{aligned}&[\xi, \xi] : [\xi, \eta] : [\xi, \zeta] : [\xi, \alpha] : [\xi, \beta]\\ =\ &[\eta, \xi] : [\eta, \eta] : [\eta, \zeta] : [\eta, \alpha] : [\eta, \beta]\\ =\ &[\zeta, \xi] : [\zeta, \eta] : [\zeta, \zeta] : [\zeta, \alpha] : [\zeta, \beta]\\ =\ &[\alpha, \xi] : [\alpha, \eta] : [\alpha, \zeta] : [\alpha, \alpha] : [\alpha, \beta]\\ =\ &[\beta, \xi] : [\beta, \eta] : [\beta, \zeta] : [\beta, \alpha] : [\beta, \beta] \quad . \quad . \quad . \quad . \quad . \quad . \quad (48).\end{aligned}$$

Art. 7.—*Examination of the Form of the Discriminant, and Calculation of its Differential Coefficients of the First and Second Orders.*

Let a_1, b_1; a_2, b_2; . . . be the common roots of (4) and (5), and let it be supposed, in the first instance, that at points in the loci considered these sets of common roots are all *distinct.*

Then if

$$\Delta = \mathrm{A}f(x, y, z, a_1, b_1)\, f(x, y, z, a_2, b_2) \ldots \qquad (49),$$

where A is a factor introduced to make the discriminant of the proper order and weight, the result of eliminating a and b between (3), (4), and (5) is

$$\Delta = 0.$$

Writing for brevity

$$\Delta = \mathrm{R}f(x, y, z, a_1, b_1) = \mathrm{R}f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$

$$\frac{\partial\Delta}{\partial x} = \frac{\partial \mathrm{R}}{\partial x} f + \mathrm{R}\left(\frac{\mathrm{D}f}{\mathrm{D}x} + \frac{\mathrm{D}f}{\mathrm{D}a_1}\frac{\partial a_1}{\partial x} + \frac{\mathrm{D}f}{\mathrm{D}b_1}\frac{\partial b_1}{\partial x}\right).$$

To determine $\partial a_1/\partial x$, $\partial b_1/\partial x$ there are the equations

$$\frac{\mathrm{D}f}{\mathrm{D}a_1} = 0, \quad \frac{\mathrm{D}f}{\mathrm{D}b_1} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51).$$

These give

$$\left.\begin{aligned}[x, a_1] + [a_1, a_1]\frac{\partial a_1}{\partial x} + [a_1, b_1]\frac{\partial b_1}{\partial x} = 0 \\ [x, b_1] + [a_1, b_1]\frac{\partial a_1}{\partial x} + [b_1, b_1]\frac{\partial b_1}{\partial x} = 0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (52).$$

Similar equations exist for finding

$$\frac{\partial a_1}{\partial y}, \frac{\partial b_1}{\partial y} \quad \text{and} \quad \frac{\partial a_1}{\partial z}, \frac{\partial b_1}{\partial z}.$$

These, however, are not yet required, it being seen at once that, in general,

$$\frac{\partial \Delta}{\partial x} = \frac{\partial \mathrm{R}}{\partial x} f + \mathrm{R}\frac{\mathrm{D}f}{\mathrm{D}x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53).$$

[For a case of exception, in which $\frac{\mathrm{D}f}{\mathrm{D}a_1}\frac{\partial a_1}{\partial x} + \frac{\mathrm{D}f}{\mathrm{D}b_1}\frac{\partial b_1}{\partial x}$ does not vanish, see Art. 19, Ex. 11, E.]

Differentiating again with regard to x,

$$\frac{\partial^2 \Delta}{\partial x^2} = \frac{\partial^2 \mathrm{R}}{\partial x^2} f + 2\frac{\partial \mathrm{R}}{\partial x}\frac{\mathrm{D}f}{\mathrm{D}x} + \mathrm{R}\left[\frac{\mathrm{D}^2 f}{\mathrm{D}x^2} + \frac{\mathrm{D}^2 f}{\mathrm{D}x\,\mathrm{D}a_1}\frac{\partial a_1}{\partial x} + \frac{\mathrm{D}^2 f}{\mathrm{D}x\,\mathrm{D}b_1}\frac{\partial b_1}{\partial x}\right] \quad . \quad . \quad (54),$$

$$\frac{\partial^2 \Delta}{\partial x\,\partial y} = \frac{\partial^2 \mathrm{R}}{\partial x\,\partial y} f + \frac{\partial \mathrm{R}}{\partial x}\frac{\mathrm{D}f}{\mathrm{D}y} + \frac{\partial \mathrm{R}}{\mathrm{D}y}\frac{\mathrm{D}f}{\mathrm{D}x} + \mathrm{R}\left[\frac{\mathrm{D}^2 f}{\mathrm{D}x\,\mathrm{D}y} + \frac{\mathrm{D}^2 f}{\mathrm{D}x\,\mathrm{D}a_1}\frac{\partial a_1}{\partial y} + \frac{\mathrm{D}^2 f}{\mathrm{D}x\,\mathrm{D}b_1}\frac{\partial b_1}{\partial y}\right] \quad . \quad (55).$$

Hence, by (52),

$$\frac{\partial^2 \Delta}{\partial x^2} = \frac{\partial^2 \mathrm{R}}{\partial x^2} f + 2\frac{\partial \mathrm{R}}{\partial x}\frac{\mathrm{D}f}{\mathrm{D}x} + \mathrm{R}\,\frac{\mathrm{D}\left[\frac{\mathrm{D}f}{\mathrm{D}x}, \frac{\mathrm{D}f}{\mathrm{D}a_1}, \frac{\mathrm{D}f}{\mathrm{D}b_1}\right]}{\mathrm{D}[x, a_1, b_1]} \Bigg/ \frac{\mathrm{D}\left[\frac{\mathrm{D}f}{\mathrm{D}a_1}, \frac{\mathrm{D}f}{\mathrm{D}b_1}\right]}{\mathrm{D}[a_1, b_1]} \quad . \quad . \quad (56),$$

$$\frac{\partial^2 \Delta}{\partial x\,dy} = \frac{\partial^2 \mathrm{R}}{\partial x\,\partial y} f + \frac{\partial \mathrm{R}}{\partial x}\frac{\mathrm{D}f}{\mathrm{D}y} + \frac{\partial \mathrm{R}}{\partial y}\frac{\mathrm{D}f}{\mathrm{D}x} + \mathrm{R}\,\frac{\mathrm{D}\left[\frac{\mathrm{D}f}{\mathrm{D}y}, \frac{\mathrm{D}f}{\mathrm{D}a_1}, \frac{\mathrm{D}f}{\mathrm{D}b_1}\right]}{\mathrm{D}[x, a_1, b_1]} \Bigg/ \frac{\mathrm{D}\left[\frac{\mathrm{D}f}{\mathrm{D}a_1}, \frac{\mathrm{D}f}{\mathrm{D}b_1}\right]}{\mathrm{D}[a_1, b_1]} \quad . \quad (57).$$

Art. 8.—*Proof of the Envelope Property.*

Let ξ, η, ζ be a point on the locus of ultimate intersections, and let the values of a, b satisfying the equations (3), (4), (5) when $x = \xi$, $y = \eta$, $z = \zeta$, be α, β. [It will be supposed first of all, that only one value of a exists, viz., α, and only one value of b exists, viz., β. But the following cases will afterwards be noticed, viz., (1) where more than one system of distinct values of a, b exist, and (2) the particular case of the preceding in which two systems of values of a, b exist which coincide.]

In this case equations (11), (25), (26) are satisfied.

Hence when $x = \xi$, $y = \eta$, $z = \zeta$,

$$\Delta = 0$$

and

$$\frac{\partial \Delta}{\partial x} \quad \text{becomes} \quad \left[R \frac{Df}{Dx} \right]_{\substack{x = \xi \\ y = \eta \\ z = \zeta}}.$$

Hence when $x = \xi$, $y = \eta$, $z = \zeta$,

$$\frac{\partial \Delta}{\partial x} \Big/ \frac{Df}{Dx} = \frac{\partial \Delta}{\partial y} \Big/ \frac{Df}{Dy} = \frac{\partial \Delta}{\partial z} \Big/ \frac{Df}{Dz} \quad \ldots \ldots \quad (58).$$

Hence the tangent planes to $\Delta = 0$, $f(x, y, z, \alpha, \beta) = 0$, coincide at ξ, η, ζ. This proves that the locus of ultimate intersections is generally an envelope.

It should be noticed that the proof shows that the locus of ultimate intersections touches in general at each point on it *one* of the infinite number of surfaces of the system passing through that point. This will be referred to in future, to distinguish it from more complicated cases, as a case of the *ordinary* envelope.

It should also be observed that the above conclusion cannot be drawn if $x = \xi$, $y = \eta$, $z = \zeta$ make $Df/Dx = 0$, $Df/Dy = 0$, $Df/Dz = 0$.

Hence the investigation itself suggests the examination of the case in which a locus of singular points exists.

Art. 9.—*To prove that if* E = 0 *be the equation of the Envelope Locus,* Δ *contains* E *as a factor once and once only in general.*

(A). If $x = \xi$, $y = \eta$, $z = \zeta$ be a point on the envelope locus, then suppose that the values of a, b satisfying (11), (25), (26), are α, β.

Then one of the systems of values of a, b satisfying (4) and (5), must become equal to α, β when $x = \xi$, $y = \eta$, $z = \zeta$.

Suppose that a_1 becomes α, b_1 becomes β.

Hence Δ becomes $R'f(\xi, \eta, \zeta, \alpha, \beta)$, where R' is what R becomes, and therefore Δ vanishes.

Hence by Art. 1, Preliminary Theorem B, Δ contains E as a factor.

Further, Δ does not contain E more than once as a factor in general, for the value of $\partial \Delta / \partial x$ given by (53) does not in general vanish. But it would vanish if Δ contained a power of E above the first as a factor: for suppose $\Delta = E^m . \psi$, where m is a positive integer greater than unity, and ψ some rational integral function of x, y, z.

Therefore

$$\frac{\partial \Delta}{\partial x} = m\mathrm{E}^{m-1}\frac{\partial \mathrm{E}}{\partial x}\psi + \mathrm{E}^m\frac{\partial \psi}{\partial x}.$$

Hence when $x = \xi$, $y = \eta$, $z = \zeta$, $\partial \Delta/\partial x = 0$.

Consequently, Δ contains E once, and once only in general as a factor.

(B). It is necessary to examine the cases of exception.

(i). If equations (11), (25), (26) are satisfied by more than one set of distinct values of a, b; take, for example, the case where there are two sets of solutions, α_1, β_1; α_2, β_2.

Let a_1, b_1, a_2, b_2 become α_1, β_1, α_2, β_2 respectively, when $x = \xi$, $y = \eta$, $z = \zeta$.

Putting

$$\Delta = \mathrm{R}'f(x, y, z, a_1, b_1)\, f(x, y, z, a_2, b_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (59),$$

$$\frac{\partial\Delta}{\partial x} = \frac{\partial \mathrm{R}'}{\partial x} f(x, y, z, a_1, b_1,)\, f(x, y, z, a_2, b_2) + \mathrm{R}'\frac{\mathrm{D}f(x, y, z, a_1, b_1)}{\mathrm{D}x} f(x, y, z, a_2, b_2)$$

$$+ \mathrm{R}'f(x, y, z, a_1, b_1)\frac{\mathrm{D}f(x, y, z, a_2, b_2)}{\mathrm{D}x} \quad . \quad . \quad . \quad (60).$$

Now, when $x = \xi$, $y = \eta$, $z = \zeta$,

$f(x, y, z, a_1, b_1)$ becomes $f(\xi, \eta, \zeta, \alpha_1, \beta_1)$ and vanishes,
$f(x, y, z, a_2, b_2)$ becomes $f(\xi, \eta, \zeta, \alpha_2, \beta_2)$ and vanishes.

Hence $\partial\Delta/\partial x$ vanishes. Similarly $\partial\Delta/\partial y$, $\partial\Delta/\partial z$ vanish. Therefore Δ contains E^2 as a factor.

Similarly if there be p distinct sets of values of a, b satisfying (11), (25), (26), it can be shown that all the partial differential coefficients of Δ up to the $(p-1)^{\text{th}}$ order vanish. Hence Δ will contain E^p as a factor. (See examples 4 (C.), 5 (C. ii.), 6 (C.) in Arts. 10, 11, 12 respectively.)

(ii.) The case in which two of the systems of values of the parameters satisfying equations (11), (25), (26) coincide, is dealt with in Arts. (12)–(25). The case, in which more than two systems of values of the parameters satisfying equations (11), (25), (26) coincide, may be treated in a similar manner.

Example 2.—*Ordinary Envelope.*

Let the surfaces be

$$z + (x - a)(y - b) = 0.$$

(A.) *The Discriminant.*

$$\Delta = z.$$

(B.) *The Envelope Locus is* $z = 0$.

Every point on $z = 0$ is the point of contact of one only of the surfaces. Hence z occurs as a factor once only on the discriminant.

It may be noticed that $z = 0$ touches each of the surfaces at one point only, $x = a$, $y = b$. This result may be compared with the next example.

Example 3.—*Ordinary Envelope.*

Let the surfaces be

$$z + (x - a)(x^2 + y^2 - b) = 0.$$

(A.) *The Discriminant.*

$$\Delta = z.$$

(B.) *The Envelope Locus is* $z = 0$.

Every point on $z = 0$ is the point of contact of one only of the surfaces. Hence z occurs as a factor once only in the discriminant.

It may be noticed that $z = 0$ touches each of the surfaces at two points, viz., $x = a$, $y = \pm \sqrt{(b - a^2)}$.

Art. 10.—*To prove that if* $C = 0$ *be the equation of the Conic Node Locus,* Δ *contains* C^2 *as a factor in general.*

Let ξ, η, ζ be a point on the conic node locus, then equations (11)-(14) are satisfied.

Hence, by (50), the substitutions $x = \xi$, $y = \eta$, $z = \zeta$, make $\Delta = 0$; and, by (53), they also make $\partial\Delta/\partial x = 0$.

By symmetry they also make $\partial\Delta/\partial y = 0$, $\partial\Delta/\partial z = 0$.

Hence, by Art. 1, Preliminary Theorem B, Δ must contain C^2 as a factor.

Example 4.—*Locus of Conic Nodes.*

Let the surfaces be

$$\alpha (x - a)^3 + \beta (y - b)^3 + 6m (x - a)(y - b) + \gamma z^2 = 0,$$

where α, β, γ, m are fixed constants, a, b are the arbitrary parameters.

(A.) *The Discriminant.*

To eliminate a and b between $f = 0$, $Df/Da = 0$, $Df/Db = 0$, is, in this case, the same as eliminating $x - a$, $y - b$ between

$$f = 0, \quad \frac{Df}{D(x - a)} = 0, \quad \frac{Df}{D(y - b)} = 0;$$

i.e., making the equation homogeneous by putting

$$x - a = X/Z, \quad y - b = Y/Z,$$

it is necessary to find the discriminant of

$$\alpha X^3 + \beta Y^3 + 6mXYZ + \gamma z^2 Z^3 = 0.$$

The invariants will in this, and in several of the examples which follow, be calculated from the results given in SALMON's 'Higher Plane Curves,' Second Edition, Arts. 217–224.

The invariant $S = m\alpha\beta\gamma z^2 - m^4$.

The invariant $T = \alpha^2\beta^2\gamma^2 z^4 - 20\, m^3\alpha\beta\gamma z^2 - 8m^6$.

Hence $\Delta = T^2 + 64S^3 = \alpha\beta\gamma z^2 (\alpha\beta\gamma z^2 + 8m^3)^3$.

(B.) *The Conic Node Locus is* $z = 0$.

Transforming the equation to the new origin $a, b, 0$, the lowest terms are of the second degree.

Hence the new origin is a conic node on the surface.

Hence $z = 0$ is the conic node locus.

Hence Δ contains z^2 as a factor.

(C.) *Three non-consecutive Surfaces of the System touch each of the planes* $z = \pm (- 8m^3/\alpha\beta\gamma)^{1/2}$ *at each point.*

To prove this, the tangent planes to the surfaces which are parallel to the plane $z = 0$ will be found.

The tangent plane to the surface

$$\alpha (x - a)^3 + \beta (y - b)^3 + 6m (x - a) (y - b) + \gamma z^2 = 0$$

at ξ, η, ζ is

$$(x - \xi) [3\alpha (\xi - a)^2 + 6m (\eta - b)] + (y - \eta) [3\beta (\eta - b)^2 + 6m (\xi - a)] + (z - \zeta)\, 2\gamma\zeta = 0.$$

If it be parallel to $z = 0$, the coefficients of x and y must vanish, but the coefficient of z must not vanish.

Therefore,

$$\alpha (\xi - a)^2 + 2m (\eta - b) = 0 \quad \quad (61),$$

$$\beta (\eta - b)^2 + 2m (\xi - a) = 0 \quad \quad (62).$$

From these, and from the condition that ξ, η, ζ lies on the surface,

$$2m (\xi - a) (\eta - b) + \gamma\zeta^2 = 0 \quad \quad (63).$$

If these be satisfied, and ζ do not vanish, the tangent plane is $z = \zeta$.

The solutions of (61), (62), (63) are

$$\xi - a = 0, \quad \eta - b = 0, \quad \zeta = 0 \quad \quad (64),$$

$$\xi - a = - 2m\alpha^{-2/3}\beta^{-1/3}, \quad \eta - b = - 2m\alpha^{-1/3}\beta^{-2/3}, \quad \zeta = \pm (- 8m^3/\alpha\beta\gamma)^{1/2} \quad (65),$$

$$\xi - a = - 2m\omega\alpha^{-2/3}\beta^{-1/3}, \quad \eta - b = - 2m\omega^2\alpha^{-1/3}\beta^{-2/3}, \quad \zeta = \pm (- 8m^3/\alpha\beta\gamma)^{1/2} \quad (66),$$

$$\xi - a = - 2m\omega^2\alpha^{-2/3}\beta^{-1/3}, \quad \eta - b = - 2m\omega\alpha^{-1/3}\beta^{-2/3}, \quad \zeta = \pm (- 8m^3/\alpha\beta\gamma)^{1/2} \quad (67),$$

where ω is an imaginary cube root of unity.

The solution (64) corresponds to the locus of conic nodes.

In the case of the solutions (65)-(67) the tangent plane at ξ, η, ζ is $z = \zeta$.

Hence either of the planes $z = \pm(-8m^3/\alpha\beta\gamma)^{1/2}$ touches at any given point on it three of the surfaces of the system, viz., those whose parameters are given by the equations

$$a = \xi + 2m\alpha^{-2/3}\beta^{-1/3}, \quad b = \eta + 2m\alpha^{-1/3}\beta^{-2/3};$$
$$a = \xi + 2m\omega\alpha^{-2/3}\beta^{-1/3}, \quad b = \eta + 2m\omega^2\alpha^{-1/3}\beta^{-2/3};$$
$$a = \xi + 2m\omega^2\alpha^{-2/3}\beta^{-1/3}, \quad b = \eta + 2m\omega\alpha^{-1/3}\beta^{-2/3}$$

Hence by Art. 9, B (i.), each of the factors

$$z \pm (-8m^3/\alpha\beta\gamma)^{1/2}$$

may be expected to occur three times in the discriminant.

This accounts for the presence of the factors

$$\{z + (-8m^3/\alpha\beta\gamma)^{1/2}\}^3 \cdot \{z - (-8m^3/\alpha\beta\gamma)^{1/2}\}^3$$

i.e.,

$$(z^2 + 8m^3/\alpha\beta\gamma)^3$$

in the discriminant.

Art. 11.—*To prove that if $B = 0$ be the equation of the Biplanar Node Locus, Δ contains B^3 as a factor in general.*

Let ξ, η, ζ be a point on the biplanar node locus. Then equations (11)-(14), (41), (42) are satisfied.

The argument of the preceding article applies so far as Δ and its first differential coefficients are concerned.

But further the values of $\partial^2\Delta/\partial x^2$, $\partial^2\Delta/\partial x\,\partial y$, given by (56), (57) vanish, in virtue of the above mentioned equations, except in the case (to be considered presently) where the substitutions $x = \xi$, $y = \eta$, $z = \zeta$ make

$$\frac{D\left[\dfrac{Df}{Da_1}, \dfrac{Df}{Db_1}\right]}{D[a_1, b_1]} = 0,$$

i.e.,

$$\frac{D[[\alpha], [\beta]]}{D[\alpha, \beta]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68).$$

From the symmetry of the variables it follows that all the other second differential coefficients of Δ also vanish when $x = \xi$, $y = \eta$, $z = \zeta$ (or the same results follow by Art. 1, Preliminary Theorem A).

Hence by Art. 1, Preliminary Theorem B, it follows that Δ contains B^3 as a factor.

Example 5.—*Locus of Biplanar Nodes.*

Let the surfaces be

$$\alpha (x - a)^3 + \alpha (y - b)^3 + 6m (x - a - z) (y - b - z) = 0,$$

where α, m are fixed constants; a, b the arbitrary parameters.

(A.) *The Discriminant.*

With the notation of the last article, the discriminant is the same as that of the equation

$$\alpha X^3 + \alpha Y^3 + 6m (X - zZ) (Y - zZ) Z = 0.$$

Therefore

$$S = 2m^2\alpha^2z^2 - m^4.$$

$$T = 36m^2\alpha^4z^4 - 64m^3\alpha^3z^3 + 24m^4\alpha^2z^2 - 8m^6.$$

$$\Delta = 16m^4\alpha^3z^3 (9\alpha z + 4m) (3\alpha^2z^2 - 6m\alpha z + 4m^2)^2.$$

(B.) *The Biplanar Node Locus is* $z = 0$.

Transforming the equation by the substitutions $x = a + X$, $y = b + Y$, $z = Z$, it becomes

$$\alpha X^3 + \alpha Y^3 + 6m (X - Z) (Y - Z) = 0.$$

Hence the new origin is a biplanar node on the surface. Hence $z = 0$ is the locus of biplanar nodes.

(C.) (i.) *The Ordinary Envelope is* $9\alpha z + 4m = 0$.

(ii.) *The Envelope such that every point on it is the point of contact of two non-consecutive Surfaces is* $3\alpha^2z^2 - 6m\alpha z + 4m^2 = 0$.

To prove these statements it is necessary to find the tangent planes parallel to the plane $z = 0$.

Hence it is necessary to have

$$f = 0, \quad Df/Dx = 0, \quad Df/Dy = 0, \quad Df/Dz \neq 0,$$

i.e.,

$$\alpha (x - a)^3 + \alpha (y - b)^3 + 6m (x - a - z) (y - b - z) = 0 \quad . \quad . \quad (69).$$

$$\alpha (x - a)^2 + 2m (y - b - z) = 0 \quad . \quad . \quad (70).$$

$$\alpha (y - b)^2 + 2m (x - a - z) = 0 \quad . \quad . \quad (71).$$

$$x - a - z + y - b - z \neq 0 \quad . \quad . \quad (72).$$

From (70) and (71) either

$$x - a = y - b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73),$$

or

$$\alpha(x - a) + \alpha(y - b) = 2m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74).$$

(i.) Taking (73), and eliminating z and $(y - b)$ from it and (69), (70), it follows that

$$(x - a)^3 (x - a + 4m/3\alpha) = 0.$$

Now, if $x - a = 0$, then $y - b = 0$ by (73), and $z = 0$ by (69).

Hence (72) is not satisfied. This solution corresponds to the biplanar node locus.

But if

$$x - a = -\ 4m/3\alpha,$$

$$y - b = -\ 4m/3\alpha \text{ by } (73),$$

and

$$z = -\ 4m/9\alpha \text{ by } (70).$$

These values satisfy (69)–(72).

Hence $z = -\ 4m/9\alpha$ is an envelope. Each point $\xi, \eta, -\ 4m/9\alpha$ on it is the point of contact of one surface of the system whose parameters are

$$a = \xi + 4m/3\alpha, \quad b = \eta + 4m/3\alpha.$$

Hence $9\alpha z + 4m = 0$ is an ordinary envelope.

(ii.) Taking (74), and eliminating $(x - a)$ and $(y - b)$ from it and (69), (70), it follows that

$$3\alpha^2 z^2 - 6m\alpha z + 4m^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75).$$

The corresponding values of $(x - a)$, $(y - b)$ are determined by (74) and

$$\alpha(x - a)^2 - 2m(x - a) - 2mz + 4m^2/\alpha = 0.$$

These values satisfy (69)–(73).

Hence each point ξ, η, ζ on the imaginary locus (75) is the point of contact of two surfaces of the system, whose parameters a, b are determined by the equations

$$\begin{aligned} \alpha^2(\xi - a)^2 - 2m\alpha(\xi - a) - 2m\alpha\zeta + 4m^2 &= 0, \\ \alpha(\xi - a) + \alpha(\eta - b) - 2m &= 0, \end{aligned}$$

where ζ is one of the roots of (75).

This accounts for the factor

$$(3\alpha^2 z^2 - 6m\alpha z + 4m^2)^2$$

in the discriminant.

Art. 12.—*To prove that if* U = 0 *be the equation of the Locus of Uniplanar Nodes,* Δ *contains* U^6 *as a factor in general.*

(A.) Amongst the conditions satisfied at every point of the uniplanar node locus, will be found the following, see the ratios (48) :—

$$[\alpha, \alpha][\beta, \beta] - [\alpha, \beta]^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76).$$

Now, by Art. 1, Preliminary Theorem C, this means that the equations

$$Df/Da = 0, \qquad Df/Db = 0$$

are satisfied by two systems of values of the parameters which become equal when $x = \xi$, $y = \eta$, $z = \zeta$, the coordinates of a point on the uniplanar node locus.

[It must be remembered that the theorem has to be specially interpreted for the case in which Df/Da, Df/Db are both of the first degree in a, b, *i.e.*, for the case in which f is of the second degree in a, b. This is done in Section IV.].

Now this is the case previously reserved in Art. 7, Art. 9 (the second case of exception), and Art. 11 (condition (68)).

As there are, in this case, two equal values of each of the parameters, a, b, it may be expected that there will be two (not necessarily equal) values of $\partial a/\partial x$, $\partial b/\partial x$.

It will appear presently that in some cases $\partial a/\partial x$, $\partial b/\partial x$ may become infinite, but this is not the case for the uniplanar node locus, in which the values of $\partial a_1/\partial x$, $\partial b_1/\partial x$ as given by (52), become indeterminate, because the conditions (48) are satisfied.

Differentiating (52) again with regard to x, it follows that

$$[x, x, a_1] + 2[x, a_1, a_1]\frac{\partial a_1}{\partial x} + 2[x, a_1, b_1]\frac{\partial b_1}{\partial x} + [a_1, a_1, a_1]\left(\frac{\partial a_1}{\partial x}\right)^2$$

$$+ 2[a_1, a_1, b_1]\left(\frac{\partial a_1}{\partial x}\right)\left(\frac{\partial b_1}{\partial x}\right) + [a_1, b_1, b_1]\left(\frac{\partial b_1}{\partial x}\right)^2 + [a_1, a_1]\frac{\partial^2 a_1}{\partial x^2} + [a_1, b_1]\frac{\partial^2 b_1}{\partial x^2} = 0 \quad . \quad (77),$$

$$[x, x, b_1] + 2[x, a_1, b_1]\frac{\partial a_1}{\partial x} + 2[x, b_1, b_1]\frac{\partial b_1}{\partial x} + [a_1, a_1, b_1]\left(\frac{\partial a_1}{\partial x}\right)^2$$

$$+ 2[a_1, b_1, b_1]\left(\frac{\partial a_1}{\partial x}\right)\left(\frac{\partial b_1}{\partial x}\right) + [b_1, b_1, b_1]\left(\frac{\partial b_1}{\partial x}\right)^2 + [a_1, b_1]\frac{\partial^2 a_1}{\partial x^2} + [b_1, b_1]\frac{\partial^2 b_1}{\partial x^2} = 0 \quad . \quad (78).$$

Multiplying (77) by $[b_1, b_1]$, (78) by $[a_1, b_1]$, subtracting, putting $x = \xi$, $y = \eta$, $z = \zeta$, and therefore $a_1 = \alpha$, $b_1 = \beta$, and using (76), it follows that

$$
\begin{aligned}
&[\xi, \xi, \alpha][\beta, \beta] - [\xi, \xi, \beta][\alpha, \beta] \\
&\quad + 2\frac{\partial\alpha}{\partial\xi}\{[\xi, \alpha, \alpha][\beta, \beta] - [\xi, \alpha, \beta][\alpha, \beta]\} + 2\frac{\partial\beta}{\partial\xi}\{[\xi, \alpha, \beta][\beta, \beta] - [\xi, \beta, \beta][\alpha, \beta]\} \\
&\quad + \left(\frac{\partial\alpha}{\partial\xi}\right)^2\{[\alpha, \alpha, \alpha][\beta, \beta] - [\alpha, \alpha, \beta][\alpha, \beta]\} \\
&\quad + 2\left(\frac{\partial\alpha}{\partial\xi}\right)\left(\frac{\partial\beta}{\partial\xi}\right)\{[\alpha, \alpha, \beta][\beta, \beta] - [\alpha, \beta, \beta][\alpha, \beta]\} \\
&\quad + \left(\frac{\partial\beta}{\partial\xi}\right)^2\{[\alpha, \beta, \beta][\beta, \beta] - [\beta, \beta, \beta][\alpha, \beta]\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (79),
\end{aligned}
$$

and this equation with either of the equations derived from (52) by changing therein x into ξ, y into η, z into ζ, and therefore a_1 into α, b_1 into β, determines in general two values for $\partial\alpha/\partial\xi$, and two corresponding values for $\partial\beta/\partial\xi$.

The second of equations (52) gives

$$
[\beta, \xi] + [\alpha, \beta]\frac{\partial\alpha}{\partial\xi} + [\beta, \beta]\frac{\partial\beta}{\partial\xi} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (80).
$$

Eliminating $\partial\beta/\partial\xi$ between (79) and (80), it follows that

$$
\begin{aligned}
&\left(\frac{\partial\alpha}{\partial\xi}\right)^2\Big[[\alpha,\alpha,\alpha][\beta,\beta]^3 - 3[\alpha,\alpha,\beta][\beta,\beta]^2[\alpha,\beta] + 3[\alpha,\beta,\beta][\beta,\beta][\alpha,\beta]^2 - [\beta,\beta,\beta][\alpha,\beta]^3\Big] \\
&+2\frac{\partial\alpha}{\partial\xi}\left[\begin{array}{l}[\xi, \alpha, \alpha][\beta, \beta]^3 - 2[\xi, \alpha, \beta][\beta, \beta]^2[\alpha, \beta] + [\xi, \beta, \beta][\beta, \beta][\alpha, \beta]^2 \\ -[\alpha, \alpha, \beta][\beta, \beta]^2[\beta, \xi] + 2[\alpha, \beta, \beta][\beta, \beta][\alpha, \beta][\beta, \xi] - [\beta, \beta, \beta][\alpha, \beta]^2[\beta, \xi]\end{array}\right] \\
&+ \left[\begin{array}{l}\quad\ [\xi, \xi, \alpha][\beta, \beta]^3 - [\xi, \xi, \beta][\beta, \beta]^2[\alpha, \beta] \\ -2[\xi, \alpha, \beta][\xi, \beta][\beta, \beta]^2 + 2[\xi, \beta, \beta][\xi, \beta][\beta, \beta][\alpha, \beta] \\ +\ [\xi, \beta]^2[\alpha, \beta, \beta][\beta, \beta] - [\xi, \beta]^2[\beta, \beta, \beta][\alpha, \beta]\end{array}\right] = 0 \quad . \quad . \quad . \quad . \quad (81).
\end{aligned}
$$

This is in general a quadratic for $\partial\alpha/\partial\xi$.

The two corresponding values of $\partial\beta/\partial\xi$ are given by (80).

The case of exception, when the quadratic for $\partial\alpha/\partial\xi$ reduces to an equation of the first degree, viz., when

$$
[\alpha, \alpha, \alpha][\beta, \beta]^3 - 3[\alpha, \alpha, \beta][\beta, \beta]^2[\alpha, \beta] + 3[\alpha, \beta, \beta][\beta, \beta][\alpha, \beta]^2 - [\beta, \beta, \beta][\alpha, \beta]^3 = 0,
$$

will now be considered.

(B.) The meaning of the condition may be determined by means of Art. 1, Preiminary Theorem D.

Put therein $\phi(u, v) = \mathrm{D}f/\mathrm{D}u$, $\psi(u, v) = \mathrm{D}f/\mathrm{D}v$.

Then the condition of Preliminary Theorem C, which also holds, is

$$\frac{\mathrm{D}^2 f}{\mathrm{D}u^2}\frac{\mathrm{D}^2 f}{\mathrm{D}v^2} - \left(\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v}\right)^2 = 0.$$

The condition of Preliminary Theorem D is equivalent to

$$\left[\frac{\mathrm{D}^3 f}{\mathrm{D}u^3}\left(\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v}\right)^2 - 2\frac{\mathrm{D}^3 f}{\mathrm{D}u^2\,\mathrm{D}v}\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v}\frac{\mathrm{D}^2 f}{\mathrm{D}u^2} + \frac{\mathrm{D}^3 f}{\mathrm{D}u\,\mathrm{D}v^2}\left(\frac{\mathrm{D}^2 f}{\mathrm{D}u^2}\right)^2\right]\left(\frac{\mathrm{D}^2 f}{\mathrm{D}v^2}\right)^3$$
$$= \left[\frac{\mathrm{D}^3 f}{\mathrm{D}u^2\,\mathrm{D}v}\left(\frac{\mathrm{D}^2 f}{\mathrm{D}v^2}\right)^2 - 2\frac{\mathrm{D}^3 f}{\mathrm{D}u\,\mathrm{D}v^2}\frac{\mathrm{D}^2 f}{\mathrm{D}v^2}\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v} + \frac{\mathrm{D}^3 f}{\mathrm{D}v^3}\left(\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v}\right)^2\right]\left(\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v}\right)^3.$$

Dividing out by $(\mathrm{D}^2 f/\mathrm{D}u\,\mathrm{D}v)^2$ the last equation becomes, by the preceding,

$$\frac{\mathrm{D}^3 f}{\mathrm{D}u^3}\left(\frac{\mathrm{D}^2 f}{\mathrm{D}v^2}\right)^3 - 3\frac{\mathrm{D}^3 f}{\mathrm{D}u^2\,\mathrm{D}v}\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v}\left(\frac{\mathrm{D}^2 f}{\mathrm{D}v^2}\right)^2 + 3\frac{\mathrm{D}^3 f}{\mathrm{D}u\,\mathrm{D}v^2}\left(\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v}\right)^2\frac{\mathrm{D}^2 f}{\mathrm{D}v^2} - \frac{\mathrm{D}^3 f}{\mathrm{D}v^3}\left(\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v}\right)^3 = 0.$$

Now in the former part of this article, α and β correspond to u and v.

Hence the condition that the equations $\mathrm{D}f/\mathrm{D}a = 0$, $\mathrm{D}f/\mathrm{D}b = 0$ may have three coinciding systems of common roots is

$$[\alpha, \alpha, \alpha][\beta, \beta]^3 - 3[\alpha, \alpha, \beta][\beta, \beta]^2[\alpha, \beta] + 3[\alpha, \beta, \beta][\beta, \beta][\alpha, \beta]^2 - [\beta, \beta, \beta][\alpha, \beta]^3 = 0.$$

This fact must be taken account of in forming the discriminant, and the whole of the work must be modified in accordance with it. But this case will not be further discussed.

(C.) It has now been shown how to determine the values of $\partial a/\partial x$, $\partial b/\partial x$, when $x = \xi$, $y = \eta$, $z = \zeta$, the coordinates of a point on the uniplanar node locus.

Now, when $x = \xi$, $y = \eta$, $z = \zeta$,

$$f(x, y, z, a_1, b_1), \quad \frac{\partial f(x, y, z, a_1, b_1)}{\partial x}$$

both vanish.

$$\frac{\partial^2 f(x, y, z, a_1, b_1)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f(x, y, z, a_1, b_1)}{\partial x}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\mathrm{D}f(x, y, z, a_1, b_1)}{\mathrm{D}x}\right)$$
$$= [x, x] + [x, a_1]\frac{\partial a_1}{\partial x} + [x, b_1]\frac{\partial b_1}{\partial x}.$$

Hence when $x = \xi$, $y = \eta$, $z = \zeta$,

$$\frac{\partial^2 f(x, y, z, a_1, b_1)}{\partial x^2} \text{ becomes } [\xi, \xi] + [\xi, \alpha]\frac{\partial \alpha}{\partial \xi} + [\xi, \beta]\frac{\partial \beta}{\partial \xi}$$

i.e.,

$$\frac{[\xi, \xi]}{[\xi, \beta]}\left\{[\xi, \beta] + [\alpha, \beta]\frac{\partial \alpha}{\partial \xi} + [\beta, \beta]\frac{\partial \beta}{\partial \xi}\right\} \text{ by (48)},$$

therefore $\partial^2 f(x, y, z, a_1, b_1)/\partial x^2$ vanishes by (80).

In like manner $\partial^2 f(x, y, z, a_2, b_2)/\partial x^2$ also vanishes when $x = \xi$, $y = \eta$, $z = \zeta$.

Now writing $\Delta = Rf_1 f_2$ for brevity, and forming all the differential coefficients up to the fifth order inclusive, each term in any of these differential coefficients is the product of terms, one of which is f_1 or f_2 or a first or second differential coefficient of f_1 or f_2.

Hence when $x = \xi$, $y = \eta$, $z = \zeta$, all the differential coefficients up to the fifth order vanish, and, therefore, by Art. 1, Preliminary Theorem B, Δ contains U^6 as a factor.

Example 6.—*Locus of Uniplanar Nodes.*

Let the surfaces be

$$\alpha (x - a)^3 + \beta (y - b)^3 + 3 [c (x - a) + gz]^2 = 0$$

where α, β, c, g are fixed constants; a, b the arbitrary parameters.

(A.) *The Discriminant.*

The discriminant is the same as that of the equation

$$\alpha X^3 + \beta Y^3 + 3Z (cX + gzZ)^2 = 0.$$

Hence

$$S = 0$$
$$T = \alpha\beta^2 g^3 z^3 (9\alpha gz - 4c^3)$$
$$\Delta = \alpha^2\beta^4 g^6 z^6 (9\alpha gz - 4c^3)^2.$$

(B.) *The Locus of Unodes is* $z = 0$.

For putting $x = a + X$, $y = b + Y$, $z = Z$, the equation becomes

$$\alpha X^3 + \beta Y^3 + 3 (cX + gZ)^2 = 0.$$

Hence the new origin is a unode. There are no other singular points on the surface.

Hence the locus of unodes is $z = 0$.

(C.) *The Envelope such that every point on it is the point of contact of two non-consecutive Surfaces is* $9\alpha gz - 4c^3 = 0$.

To prove this it is necessary to find the tangent planes parallel to $z = 0$.

Hence it is necessary to find x, y, z, so that

$$f = \alpha (x-a)^3 + \beta (y-b)^3 + 3[c(x-a)+gz]^2 = 0 \quad . \quad . \quad . \quad (82),$$

$$\tfrac{1}{3}\frac{\mathrm{D}f}{\mathrm{D}x} = \alpha (x-a)^2 + 2c[c(x-a)+gz] \qquad = 0 \quad . \quad . \quad . \quad (83),$$

$$\tfrac{1}{3}\frac{\mathrm{D}f}{\mathrm{D}y} = \beta (y-b)^2 \qquad = 0 \quad . \quad . \quad . \quad (84),$$

$$\tfrac{1}{3}\frac{\mathrm{D}f}{\mathrm{D}z} = 2g[c(x-a)+gz] \qquad \neq 0 \quad . \quad . \quad . \quad (85).$$

Therefore

$$y = b,$$
$$\alpha (x-a)^3 + 3[c(x-a)+gz]^2 = 0,$$
$$\alpha (x-a)^2 + 2c[c(x-a)+gz] = 0;$$

therefore

$$[c(x-a)+gz][c(x-a)+3gz] = 0.$$

The solution $c(x-a)+gz = 0$ is inconsistent with (85).
Hence it is necessary to take

$$c(x-a)+3gz = 0.$$

Substituting in (83),

$$\frac{gz}{c^2}(9g\alpha z - 4c^3) = 0.$$

The solution $z = 0$, gives $x = a$, $y = b$, and therefore belongs to the unode locus.
The solution $9g\alpha z - 4c^3 = 0$
gives

$$\alpha (x-a)^2 + 2c^2(x-a) + 8c^4/9\alpha = 0,$$

and therefore

$$x - a = -4c^2/3\alpha,$$

or

$$x - a = -2c^2/3\alpha.$$

Hence at every point ξ, η, $4c^3/9g\alpha$ on this locus, the locus is touched by two non-consecutive surfaces of the system, viz., those whose parameters are given by

$$a = \xi + 4c^2/3\alpha, \qquad b = \eta;$$

and

$$a = \xi + 2c^2/3\alpha, \qquad b = \eta.$$

This accounts for the factor

$$(9g\alpha z - 4c^3)^2$$

in the discriminant.

Example 7.—*Locus of Uniplanar Nodes.*

Let the surfaces be

$$\alpha (x - a)^3 + \beta (y - b)^3 - 3n (z - ax + by)^2 = 0,$$

where α, β, n are fixed constants, and a, b are the arbitrary parameters.

(A.) *The Discriminant.*

Putting $\zeta = z - x^2 + y^2$, the equation is

$$\alpha (x - a)^3 + \beta (y - b)^3 - 3n \{\zeta + x (x - a) - y (y - b)\}^2 = 0.$$

Hence the discriminant is the same as that of the equation

$$\alpha X^3 + \beta Y^3 - 3nZ (\zeta Z + xX - yY)^2 = 0.$$

Therefore

$$S = n^2 \alpha\beta xy\zeta^2$$
$$T = 9n^2\alpha^2\beta^2\zeta^4 + 4n^3\alpha\beta\zeta^3 (\beta x^3 - \alpha y^3).$$

Therefore

$$\Delta = n^4\alpha^2\beta^2\zeta^6 [\{9\alpha\beta\zeta + 4n (\beta x^3 - \alpha y^3)\}^2 + 64n^2\alpha\beta x^3y^3].$$

In order to show the way in which the factor ζ^6 arises in the discriminant, the discriminant will now be calculated.

It is known to be the result of eliminating X, Y, Z from $Df/DX = 0$, $Df/DY = 0$, $Df/DZ = 0$, *i.e.*, from

$$\alpha X^2 - 2nxZ (\zeta Z + xX - yY) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86),$$

$$\beta Y^2 + 2nyZ (\zeta Z + xX - yY) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87),$$

$$(\zeta Z + xX - yY) (3\zeta Z + xX - yY) = 0 \quad . \quad . \quad . \quad . \quad (88).$$

From (86) and (87)

$$Y = \pm X \sqrt{\left(-\frac{\alpha y}{\beta x}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89).$$

Substituting in (86)

$$\alpha X^2 - 2nxXZ \left\{x \mp y \sqrt{\left(-\frac{\alpha y}{\beta x}\right)}\right\} - 2nx\zeta Z^2 = 0.$$

Put

$$x - y \sqrt{\left(-\frac{\alpha y}{\beta x}\right)} = \xi, \quad x + y \sqrt{\left(-\frac{\alpha y}{\beta x}\right)} = \eta.$$

Then two of the values of X/Z are found from

$$\alpha X^2 - 2nx\xi XZ - 2nx\zeta Z^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90),$$

and then

$$Y = +X\sqrt{\left(-\frac{\alpha y}{\beta x}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91);$$

and the remaining two values from

$$\alpha X^2 - 2nx\eta XZ - 2nx\zeta Z^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (92),$$

and then

$$Y = -X\sqrt{\left(-\frac{\alpha y}{\beta x}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93).$$

Take the factors of the left-hand side of (88) separately, and form first of all the part of the discriminant depending on

$$\zeta Z + xX - yY.$$

Let the roots of (90) be X_1/Z_1 and X_2/Z_2; then the corresponding values of Y by (91) are

$$X_1\sqrt{\left(-\frac{\alpha y}{\beta x}\right)} \quad \text{and} \quad X_2\sqrt{\left(-\frac{\alpha y}{\beta x}\right)}$$

Hence

$$\begin{aligned}(\zeta Z_1 + xX_1 - yY_1)(\zeta Z_2 + xX_2 - yY_2) \\ &= (\zeta Z_1 + \xi X_1)(\zeta Z_2 + \xi X_2) \\ &= Z_1Z_2\left(\zeta^2 + \zeta\xi\left(\frac{X_1}{Z_1} + \frac{X_2}{Z_2}\right) + \xi^2\frac{X_1X_2}{Z_1Z_2}\right) \\ &= Z_1Z_2(\zeta^2).\end{aligned}$$

In like manner, taking the roots corresponding to (92) and (93),

$$(\zeta Z_3 + xX_3 - yY_3)(\zeta Z_4 + xX_4 - yY_4) = Z_3Z_4\zeta^2.$$

Next, taking the other factor of the left-hand side of (88), the part of the discriminant corresponding to the roots of (90) and (91) is

$$\begin{aligned}(3\zeta Z_1 + xX_1 - yY_1)(3\zeta Z_2 + xX_2 - yY_2) \\ &= (3\zeta Z_1 + \xi X_1)(3\zeta Z_2 + \xi X_2) \\ &= Z_1Z_2\left(9\zeta^2 + 3\zeta\xi\left(\frac{X_1}{Z_1} + \frac{X_2}{Z_2}\right) + \xi^2\frac{X_1X_2}{Z_1Z_2}\right) \\ &= Z_1Z_2\left(9\zeta^2 + \frac{4nx\zeta\xi^2}{\alpha}\right).\end{aligned}$$

In like manner the roots of (92) and (93) give rise to the following part of the discriminant:—

$$Z_3Z_4\left(9\zeta^2 + \frac{4nx\xi\eta^2}{\alpha}\right).$$

Hence the roots of (90) and (91) give rise to the portion

$$(Z_1Z_2)^2\left(9\zeta + \frac{4nx\xi^2}{\alpha}\right)\zeta^3,$$

and the roots of (92) and (93) to

$$(Z_3Z_4)^2\left(9\zeta + \frac{4nx\eta^2}{\alpha}\right)\zeta^3.$$

Hence the discriminant is

$$(Z_1Z_2Z_3Z_4)^2\zeta^6\left(81\zeta^2 + \frac{36nx\zeta}{\alpha}(\xi^2+\eta^2) + \frac{16n^2x^2\xi^2\eta^2}{\alpha^2}\right)$$

$$= (Z_1Z_2Z_3Z_4)^2\zeta^6\left(81\zeta^2 + \frac{72n\zeta}{\alpha\beta}(\beta x^3 - \alpha y^3) + \frac{16n^2(\beta x^3+\alpha y^3)^2}{\alpha^2\beta^2}\right)$$

$$= \left(\frac{Z_1Z_2Z_3Z_4}{\alpha\beta}\right)^2\zeta^6\left[\{9\alpha\beta\zeta + 4n(\beta x^3 - \alpha y^3)\}^2 + 64n^2\alpha\beta x^3y^3\right].$$

This agrees with the result previously stated.

Returning now to the part of the discriminant arising from the two systems of roots of (90) and (91), it will be shown that the factor ζ^3 arises entirely from one of the systems only.

Consider, in fact,

$$(\zeta Z_1 + xX_1 - yY_1)(3\zeta Z_1 + xX_1 - yY_1),$$

which is the part of the discriminant due to the system of roots X_1, Y_1, Z_1.

It is equal to

$$(\zeta Z_1 + \xi X_1)(3\zeta Z_1 + \xi X_1) = Z_1^2\{3\zeta^2 + 4\xi\zeta(X_1/Z_1) + \xi^2(X_1/Z_1)^2\}$$

where

$$\alpha(X_1/Z_1)^2 - 2nx\xi(X_1/Z_1) - 2nx\zeta = 0.$$

Therefore

$$(\zeta Z_1 + xX_1 - yY_1)(3\zeta Z_1 + xX_1 - yY_1) = Z_1^2\left\{3\zeta^2 + \frac{2nx\xi^2\zeta}{\alpha} + \frac{X_1}{Z_1}\left(4\xi\zeta + \frac{2nx\xi^3}{\alpha}\right)\right\}.$$

Now,

$$\frac{X_1}{Z_1} = \frac{nx\xi}{\alpha} \pm \frac{\sqrt{(n^2x^2\xi^2 + 2nx\alpha\zeta)}}{\alpha}$$

$$= \frac{nx\xi}{\alpha} \pm \frac{nx\xi}{\alpha}\left\{1 + \frac{\alpha\zeta}{nx\xi^2} - \frac{1}{2}\frac{\alpha^2\zeta^2}{n^2x^2\xi^4} + \frac{1}{2}\frac{\alpha^3\zeta^3}{n^3x^3\xi^6} - \dots\right\}.$$

If the root corresponding to the positive sign be taken, then

$$(\zeta Z_1 + xX_1 - yY_1)(3\zeta Z_1 + xX_1 - yY_1)$$

is not divisible by ζ.

If, however, the root corresponding to the negative sign be taken, *i.e.*,

$$\frac{X_1}{Z_1} = -\left\{\frac{\zeta}{\xi} - \tfrac{1}{2}\frac{\alpha\zeta^2}{nx\xi^3} + \tfrac{1}{2}\frac{\alpha^2\zeta^3}{n^2x^2\xi^5} - \dots\right\}$$

then

$$(\zeta Z_1 + xX_1 - yY_1)(3\zeta Z_1 + xX_1 - yY_1)$$
$$= Z_1^2\left\{\frac{\alpha\zeta^3}{nx\xi^2} + \text{terms containing higher powers of } \zeta\right\}$$

Hence the factor ζ^3 arises exclusively from the substitution in f of one system of values of the parameters satisfying both the equations $Df/Da = 0$, $Df/Db = 0$.

A similar demonstration shows that the solution

$$\frac{X_3}{Z_3} = -\left\{\frac{\zeta}{\eta} - \tfrac{1}{2}\frac{\alpha\zeta^2}{nx\eta^3} + \tfrac{1}{2}\frac{\alpha^2\zeta^3}{n^2x^2\eta^5} - \dots\right\}$$

and

$$Y_3 = -X_3\sqrt{\left(-\frac{\alpha y}{\beta x}\right)},$$

will also give rise to the factor ζ^3.

Now it will be shown presently that $\zeta = 0$ is the unode locus. Hence at any point on the unode locus,

$$\frac{X_1}{Z_1} = 0,\quad \frac{Y_1}{Z_1} = 0\,;\quad \frac{X_3}{Z_3} = 0,\quad \frac{Y_3}{Z_3} = 0.$$

Hence, at such a point there are two values of $x - a$ and two of $y - b$ which vanish. Hence two systems of values of a, b, satisfying both the equations $Df/Da = 0$, $Df/Db = 0$ become equal; viz., the two values of the parameter a become equal to the x-coordinate of the point, and the two values of the parameter b become equal to the y-coordinate of the point.

(B.) *The Locus of Uniplanar Nodes is* $\zeta = 0$.

To find the singular points, it is necessary to find values of x, y, z satisfying all the equations

$$\begin{aligned}
\alpha(x-a)^3 + \beta(y-b)^3 - 3n(z - ax + by)^2 &= 0,\\
\alpha(x-a)^2 + 2na(z - ax + by) &= 0,\\
\beta(y-b)^2 - 2nb(z - ax + by) &= 0,\\
z - ax + by &= 0.
\end{aligned}$$

The only solutions of which are

$$x = a,\quad y = b,\quad z = a^2 - b^2.$$

Now, transforming the equation by means of the substitutions $x = a + X$, $y = b + Y$, $z = a^2 - b^2 + Z$, it becomes

$$\alpha X^3 + \beta Y^3 - 3n(Z - aX + bY)^2 = 0.$$

Hence, the new origin is a unode.

Hence the locus of unodes is $z = x^2 - y^2$, *i.e.*, $\zeta = 0$.

(C.) *The Envelope Locus is*

$$\{9\alpha\beta\zeta + 4n(\beta x^3 - \alpha y^3)\}^2 + 64\alpha\beta n^2 x^3 y^3 = 0 \quad . \quad . \quad . \quad . \quad (94).$$

On examining the manner in which the discriminant was formed it can be seen that the factor corresponding to the envelope focus is obtained from (86), (87), and $3\zeta Z + xX - yY = 0$.

Hence it is the result of eliminating X, Y, Z from

$$\left.\begin{array}{r} \zeta Z + xX - yY = -2\zeta Z \\ \alpha X^2 + 4nx\zeta Z^2 = 0 \\ \beta Y^2 - 4ny\zeta Z^2 = 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95).$$

To prove that (94) is an envelope locus, it is necessary to show that if ϕ be the left-hand side of (94), and $f = 0$ be the equation of the system of surfaces, then it is possible to find values of x, y, z which satisfy at the same time all the equations—

$$\left.\begin{array}{c} f = 0, \quad \phi = 0 \\ \dfrac{Df}{Dx}\Big/\dfrac{\partial\phi}{\partial x} = \dfrac{Df}{Dy}\Big/\dfrac{\partial\phi}{\partial y} = \dfrac{Df}{Dz}\Big/\dfrac{\partial\phi}{\partial z} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96).$$

Changing the independent variables from x, y, z to x, y, ζ, then, if δ denote partial differentiation when x, y, ζ are independent variables, equations (96) are equivalent to

$$f = 0, \quad \phi = 0$$

$$\frac{\delta f}{\delta x}\Big/\frac{\delta\phi}{\delta x} = \frac{\delta f}{\delta y}\Big/\frac{\delta\phi}{\delta y} = \frac{\delta f}{\delta z}\Big/\frac{\delta\phi}{\delta z},$$

i.e.

$$\frac{3\alpha(x-a)^2 - 6n(2x-a)[\zeta + (x-a)x - (y-b)y]}{24n\beta x^2[9\alpha\beta\zeta + 4n(\beta x^3 - \alpha y^3)] + 192n^2\alpha\beta x^2 y^3}$$

$$= \frac{3\beta(y-b)^2 + 6n(2y-b)[\zeta + (x-a)x - (y-b)y]}{-24n\alpha y^2[9\alpha\beta\zeta + 4n(\beta x^3 - \alpha y^3)] + 192n^2\alpha\beta x^3 y^2}$$

$$= \frac{-6n[\zeta + x(x-a) - y(y-b)]}{18\alpha\beta[9\alpha\beta\zeta + 4n(\beta x^3 - \alpha y^3)]};$$

or, putting $x - a = X/Z, \quad y - b = Y/Z$,

$$\frac{\alpha X^2 - 2n(xZ + X)(\zeta Z + xX - yY)}{24n\beta x^2[9\alpha\beta\zeta + 4n(\beta x^3 + \alpha y^3)]}$$
$$= \frac{\beta Y^2 + 2n(yZ + Y)(\zeta Z + xX - yY)}{-24n\alpha y^2[9\alpha\beta\zeta - 4n(\beta x^3 + \alpha y^3)]}$$
$$= \frac{-2nZ(\zeta Z + xX - yY)}{18\alpha\beta[9\alpha\beta\zeta + 4n(\beta x^3 - \alpha y^3)]}.$$

It will now be shown that the values of x, y, z which satisfy (95) also satisfy these equations.

For substituting from (95), these equations become

$$\frac{\alpha X^2 + 4n\zeta Z(xZ + X)}{4n\beta x^2[9\alpha\beta\zeta + 4n(\beta x^3 + \alpha y^3)]} = \frac{\beta Y^2 - 4n\zeta Z(yZ + Y)}{-4n\alpha y^2[9\alpha\beta\zeta - 4n(\beta x^3 + \alpha y^3)]}$$
$$= \frac{4n\zeta Z^2}{3\alpha\beta[9\alpha\beta\zeta + 4n(\beta x^3 - \alpha y^3)]}$$

These reduce further to

$$\frac{4n\zeta ZX}{4n\beta x^2[9\alpha\beta\zeta + 4n(\beta x^3 + \alpha y^3)]} = \frac{-4n\zeta ZY}{-4n\alpha y^2[9\alpha\beta\zeta - 4n(\beta x^3 + \alpha y^3)]}$$
$$= \frac{4n\zeta Z^2}{3\alpha\beta[9\alpha\beta\zeta + 4n(\beta x^3 - \alpha y^3)]},$$

i.e.,

$$\frac{X}{4n\beta x^2[9\alpha\beta\zeta + 4n(\beta x^3 + \alpha y^3)]} = \frac{Y}{4n\alpha y^2[9\alpha\beta\zeta - 4n(\beta x^3 + \alpha y^3)]}$$
$$= \frac{Z}{3\alpha\beta[9\alpha\beta\zeta + 4n(\beta x^3 - \alpha y^3)]}.$$

Now the relations (95) satisfy (94).

Further (94) can be written in either of the forms

$$[9\alpha\beta\zeta + 4n(\beta x^3 + \alpha y^3)]^2 = 144n\alpha^2\beta\zeta y^3,$$
$$[9\alpha\beta\zeta - 4n(\beta x^3 + \alpha y^3)]^2 = -144n\alpha\beta^2\zeta x^3.$$

Hence it is necessary to show that

$$\frac{X}{48n\alpha\beta x^2 y\sqrt{(n\beta y\zeta)}} = \frac{Y}{48n\alpha\beta xy^2\sqrt{(-n\alpha x\zeta)}} = \frac{Z}{24n\alpha\beta xy\sqrt{(-\alpha\beta xy)}},$$

i.e.,

$$X^2/Z^2 = -4xn\zeta/\alpha$$
$$Y^2/Z^2 = 4ny\zeta/\beta,$$

and these are true by (95).

Further, $\phi = 0$, $f = 0$ are both consequences of (95).

Hence all the equations (96) are satisfied by the same values of x, y, z.

SECTION III. (Arts. 13–15).—CONSIDERATION OF THE CASES RESERVED IN THE PREVIOUS SECTION, IN WHICH TWO SYSTEMS OF VALUES OF THE PARAMETERS SATISFYING THE EQUATIONS, $Df/Da = 0$, $Df/Db = 0$, COINCIDE AT A POINT ON THE LOCUS OF ULTIMATE INTERSECTIONS.

The interpretation of the condition

$$[\alpha, \alpha]\,[\beta, \beta] - [\alpha, \beta]^2 = 0,$$

which is marked (76), when the equation of the system of surfaces is of the second degree in the parameters, is different from its interpretation when it is of a higher degree.

It will be supposed, in this section, that the degree of the equation of the system of surfaces in the parameters is higher than the second.

Art. 13.—*To prove that if each Surface of the System have Stationary Contact with the Envelope, then Δ contains E^2 as a factor.*

(A.) It will be shown that when the condition (76) holds in the case of an envelope locus, the curve of intersection of the envelope with each surface of the system has a double point at the point of contact, such that the two tangents coincide. [Such contact is called stationary (see SALMON'S 'Geometry of Three Dimensions,' 3rd Edition, Arts. 204, 300).]

To prove this, it is necessary to find the direction of the tangents to the curve of intersection of the envelope and one of the surfaces of the system at the point of contact.

Let ξ, η, ζ, be a point on the envelope. Let the surface touching the envelope at this point be

$$f(x, y, z, \alpha, \beta) = 0,$$

which has been marked (9).

Then equations (11), (25), (26) hold.

Let $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$ be a point near to ξ, η, ζ, which lies on the curve of intersection of the surface (9) and the envelope.

Since it is on the envelope, it will be the point of contact of one of the surfaces of the system.

Suppose it is the point of contact of the surface (10).

Then

$$f(\xi + \delta\xi,\ \eta + \delta\eta,\ \zeta + \delta\zeta, \alpha, \beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (97),$$

and the equations obtained from (11), (25), (26) by changing ξ, η, ζ, α, β into $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$, $\alpha + \delta\alpha$, $\beta + \delta\beta$.

Then, from (97)

$$[\xi](\delta\xi)+[\eta](\delta\eta)+[\zeta](\delta\zeta)$$
$$+\tfrac{1}{2}\left\{\begin{array}{l}[\xi,\xi](\delta\xi)^2+[\eta,\eta](\delta\eta)^2+[\zeta,\zeta](\delta\zeta)^2\\ \qquad +2[\eta,\zeta](\delta\eta)(\delta\zeta)+2[\zeta,\xi](\delta\zeta)(\delta\xi)+2[\xi,\eta](\delta\xi)(\delta\eta)\end{array}\right\}$$
$$+\text{ terms of the third and higher orders}=0 \quad \ldots \quad (98),$$

and by means of the substitutions in (11), (25), (26),

$$f(\xi,\eta,\zeta,\alpha,\beta)$$
$$+[\xi](\delta\xi)+[\eta](\delta\eta)+[\zeta](\delta\zeta)+[\alpha](\delta\alpha)+[\beta](\delta\beta)$$
$$+\tfrac{1}{2}\left[\begin{array}{l}[\xi,\xi](\delta\xi)^2+[\eta,\eta](\delta\eta)^2+[\zeta,\zeta](\delta\zeta)^2\\ \qquad +2[\eta,\zeta](\delta\eta)(\delta\zeta)+2[\zeta,\xi](\delta\zeta)(\delta\xi)+2[\xi,\eta](\delta\xi)(\delta\eta)\\ +2[\xi,\alpha](\delta\xi)(\delta\alpha)+2[\eta,\alpha](\delta\eta)(\delta\alpha)+2[\zeta,\alpha](\delta\zeta)(\delta\alpha)\\ +2[\xi,\beta](\delta\xi)(\delta\beta)+2[\eta,\beta](\delta\eta)(\delta\beta)+2[\zeta,\beta](\delta\zeta)(\delta\beta)\\ +[\alpha,\alpha](\delta\alpha)^2+2[\alpha,\beta](\delta\alpha)(\delta\beta)+[\beta,\beta](\delta\beta)^2\end{array}\right]$$
$$+\text{ terms of the third and higher orders}=0 \quad \ldots \quad (99),$$

$$[\alpha]+[\alpha,\xi](\delta\xi)+[\alpha,\eta](\delta\eta)+[\alpha,\zeta](\delta\zeta)+[\alpha,\alpha](\delta\alpha)+[\alpha,\beta](\delta\beta)$$
$$+\text{ terms of the second and higher orders}=0 \quad \ldots \quad (100),$$

$$[\beta]+[\beta,\xi](\delta\xi)+[\beta,\eta](\delta\eta)+[\beta,\zeta](\delta\zeta)+[\beta,\alpha](\delta\alpha)+[\beta,\beta](\delta\beta)$$
$$+\text{ terms of the second and higher orders}=0 \quad \ldots \quad (101).$$

Making use of (11), (25), (26), (98), equations (99)–(101) become

$$[\xi,\alpha](\delta\xi)(\delta\alpha)+[\eta,\alpha](\delta\eta)(\delta\alpha)+[\zeta,\alpha](\delta\zeta)(\delta\alpha)$$
$$+[\xi,\beta](\delta\xi)(\delta\beta)+[\eta,\beta](\delta\eta)(\delta\beta)+[\zeta,\beta](\delta\zeta)(\delta\beta)$$
$$+\tfrac{1}{2}[\alpha,\alpha](\delta\alpha)^2+[\alpha,\beta](\delta\alpha)(\delta\beta)+\tfrac{1}{2}[\beta,\beta](\delta\beta)^2$$
$$+\text{ terms of the third and higher orders}=0 \quad \ldots \quad (102),$$

$$[\alpha,\xi](\delta\xi)+[\alpha,\eta](\delta\eta)+[\alpha,\zeta](\delta\zeta)+[\alpha,\alpha](\delta\alpha)+[\alpha,\beta](\delta\beta)$$
$$+\text{ terms of the second and higher orders}=0\,. \quad (103),$$

$$[\beta,\xi](\delta\xi)+[\beta,\eta](\delta\eta)+[\beta,\zeta](\delta\zeta)+[\beta,\alpha](\delta\alpha)+[\beta,\beta](\delta\beta)$$
$$+\text{ terms of the second and higher orders}=0\,. \quad (104).$$

By (102), (103), and (104) it follows that

$$\tfrac{1}{2}[\alpha, \alpha](\delta\alpha)^2 + [\alpha, \beta](\delta\alpha)(\delta\beta) + \tfrac{1}{2}[\beta, \beta](\delta\beta)^2$$
$$+ \text{ terms of the third and higher orders} = 0$$

Hence ultimately

$$[\alpha, \alpha](\delta\alpha)^2 + 2[\alpha, \beta](\delta\alpha)(\delta\beta) + [\beta, \beta](\delta\beta)^2 = 0.$$

This determines the two values of the ratio $\delta\beta/\delta\alpha$.

Then to determine $\delta\xi$, $\delta\eta$, $\delta\zeta$ there are the following equations obtained from (98), (103), (104), by retaining only the principal terms.

$$[\xi](\delta\xi) + [\eta](\delta\eta) + [\zeta](\delta\zeta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (105),$$

$$[\alpha, \xi](\delta\xi) + [\alpha, \eta](\delta\eta) + [\alpha, \zeta](\delta\zeta) + [\alpha, \alpha](\delta\alpha) + [\alpha, \beta](\delta\beta) = 0 \, . \, (106),$$

$$[\beta, \xi](\delta\xi) + [\beta, \eta](\delta\eta) + [\beta, \zeta](\delta\zeta) + [\beta, \alpha](\delta\alpha) + [\beta, \beta](\delta\beta) = 0 \, . \, (107).$$

Hence the ratios $\delta\xi : \delta\eta : \delta\zeta$ can be determined.

Hence the directions of the tangents to each of the branches of the curve of intersection of the envelope and the surface (9) can be determined.

If, now, the condition (76) hold, the two values of $\delta\beta/\delta\alpha$ become equal, and, therefore, the two tangents at the double point of the curve of intersection coincide, and therefore, the contact is stationary.

Further, because in this case the values of $\delta\beta/\delta\alpha$ both become equal to

$$-[\alpha, \alpha]/[\alpha, \beta] = -[\alpha, \beta]/[\beta, \beta],$$

therefore (106) and (107) become

$$[\alpha, \xi](\delta\xi) + [\alpha, \eta](\delta\eta) + [\alpha, \zeta](\delta\zeta) = 0,$$
$$[\beta, \xi](\delta\xi) + [\beta, \eta](\delta\eta) + [\beta, \zeta](\delta\eta) = 0.$$

From these two equations and (105) it follows that the coinciding tangents at the double point of the curve of intersection lie in the tangent planes to the surfaces $Df/D\alpha = 0$, $Df/D\beta = 0$, $f = 0$.

(B.) In this case

$$\Delta = Rf(x, y, z, a_1, b_1) f(x, y, z, a_2, b_2)$$
$$= Rf_1 f_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108).$$

$$\frac{\partial\Delta}{\partial x} = \frac{\partial R}{\partial x} f_1 f_2 + R\left(\frac{Df_1}{Dx} f_2 + f_1 \frac{Df_2}{Dx}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (109).$$

Hence $\Delta = 0$, $\partial\Delta/\partial x = 0$; for $f_1 = 0, f_2 = 0$ at every point on the envelope locus. Hence Δ contains E^2 as a factor.

Example 8.—*Envelope Locus, each Surface of the System having Stationary Contact with the Envelope.*

Let the surfaces be

$$\alpha (x-a)^3 + \beta (y-b)^3 + 3[\gamma (x-a) + \delta (y-b)]^2 + z^2 - c^2 = 0,$$

where $\alpha, \beta, \gamma, \delta, c$ are fixed constants ; a, b the arbitrary parameters.

(A.) *The Discriminant.*

It is the same as that of the equation

$$\alpha X^3 + \beta Y^3 + 3Z (\gamma X + \delta Y)^2 + (z^2 - c^2) Z^3 = 0.$$

Therefore

$$S = \alpha\beta\gamma\delta (z^2 - c^2),$$
$$T = (z^2 - c^2)[\alpha^2\beta^2 (z^2 - c^2) + 4 (\alpha\delta^3 - \beta\gamma^3)^2].$$

Therefore

$$\Delta = (z^2 - c^2)^2 [\{\alpha^2\beta^2 (z^2 - c^2) + 4 (\alpha\delta^3 - \beta\gamma^3)^2\}^2 + 64\alpha^3\beta^3\gamma^3\delta^3 (z^2 - c^2)].$$

(B.) *The Envelope, such that each Surface has Stationary Contact with it, is* $z^2 - c^2 = 0$.

Transform the equation by means of the equations $x = a + X$, $y = b + Y$, $z = \pm c + Z$, and it becomes

$$\alpha X^3 + \beta Y^3 + 3 (\gamma X + \delta Y)^2 + Z^2 \pm 2cZ = 0.$$

The tangent plane at the new origin is $Z = 0$; it cuts the surface in the curve

$$\alpha X^3 + \beta Y^3 + 3 (\gamma X + \delta Y)^2 = 0,$$

which has a cusp at the origin.

Hence the contact is stationary.

Hence the factor $(z^2 - c^2)^2$ in the discriminant is accounted for.

(C.) *The Locus*

$$\{\alpha^2\beta^2 (z^2 - c^2) + 4 (\alpha\delta^3 - \beta\gamma^3)^2\}^2 + 64\alpha^3\beta^3\gamma^3\delta^3 (z^2 - c^2) = 0$$

is an ordinary envelope.

This may be proved by finding the tangent planes parallel to the plane $z = 0$.

It is necessary to satisfy at the same time

$$\alpha (x-a)^3 + \beta (y-b)^3 + 3[\gamma (x-a) + \delta (y-b)]^2 + z^2 - c^2 = 0 \quad (110)$$

$$\left.\begin{aligned} \alpha (x-a)^2 + 2\gamma [\gamma (x-a) + \delta (y-b)] &= 0 \\ \beta (y-b)^2 + 2\delta [\gamma (x-a) + \delta (y-b)] &= 0 \\ 2z &\neq 0 \end{aligned}\right\}. \quad (111).$$

From the above

$$[\gamma(x-a)+\delta(y-b)]^2+(z^2-c^2)=0,$$

therefore

$$\gamma(x-a)+\delta(y-b)=\sqrt{(c^2-z^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (112)$$

Hence by (111) and (112)

$$x-a=\pm\sqrt{\left\{-\frac{2\gamma}{\alpha}(c^2-z^2)^{1/2}\right\}}$$

$$y-b=\pm\sqrt{\left\{-\frac{2\delta}{\beta}(c^2-z^2)^{1/2}\right\}}.$$

Substituting in (112),

$$\pm\gamma\sqrt{\left\{-\frac{2\gamma}{\alpha}(c^2-z^2)^{1/2}\right\}}\pm\delta\sqrt{\left\{-\frac{2\delta}{\beta}(c^2-z^2)^{1/2}\right\}}=(c^2-z^2)^{1/2},$$

therefore

$$\left\{-2\left(\frac{\gamma^3}{\alpha}+\frac{\delta^3}{\beta}\right)\pm 2\gamma\delta\sqrt{\frac{4\gamma\delta}{\alpha\beta}}\right\}(c^2-z^2)^{1/2}=(c^2-z^2),$$

therefore

$$-2\left(\frac{\gamma^3}{\alpha}+\frac{\delta^3}{\beta}\right)\pm 2\gamma\delta\sqrt{\frac{4\gamma\delta}{\alpha\beta}}=(c^2-z^2)^{1/2},$$

therefore

$$4(\gamma^3\beta+\delta^3\alpha)^2+16\alpha\beta\gamma^3\delta^3\pm 16(\gamma^3\beta+\delta^3\alpha)\,\alpha^{1/2}\beta^{1/2}\gamma^{3/2}\delta^{3/2}=\alpha^2\beta^2(c^2-z^2).$$

Therefore

$$[\alpha^2\beta^2(z^2-c^2)+4(\gamma^3\beta+\delta^3\alpha)^2+16\alpha\beta\gamma^3\delta^3]^2=256\alpha\beta\gamma^3\delta^3(\gamma^3\beta+\delta^3\alpha)^2.$$

This reduces to

$$[\alpha^2\beta^2(z^2-c^2)+4(\alpha\delta^3-\beta\gamma^3)^2]^2+64\alpha^3\beta^3\gamma^3\delta^3(z^2-c^2)=0.$$

This accounts for the remaining factor in the discriminant. It corresponds to an ordinary envelope.

Art. 14.—*To prove that if the Conic Node Locus be also an Envelope,* Δ *contains* C^3 *as a factor.*

(A.) It will be shown that when the condition (76) holds, in the case of a conic node locus, then the conic node locus is also an envelope.

In this case, from (28) and (29), by means of (76), it follows that the equation of the tangent plane to the conic node locus is

$$\begin{aligned}&[\beta, \alpha]\{[\alpha, \xi](X-\xi)+[\alpha, \eta](Y-\eta)+[\alpha, \zeta](Z-\zeta)\}\\ &-[\alpha, \alpha]\{[\beta, \xi](X-\xi)+[\beta, \eta](Y-\eta)+[\beta, \zeta](Z-\zeta)\}=0 \qquad (113).\end{aligned}$$

This will touch the tangent cone (33) at the conic node, if

$$\begin{vmatrix} [\xi,\xi] & [\xi,\eta] & [\xi,\zeta] & [\beta,\alpha][\alpha,\xi]-[\alpha,\alpha][\beta,\xi] \\ [\eta,\xi] & [\eta,\eta] & [\eta,\zeta] & [\beta,\alpha][\alpha,\eta]-[\alpha,\alpha][\beta,\eta] \\ [\zeta,\xi] & [\zeta,\eta] & [\zeta,\zeta] & [\beta,\alpha][\alpha,\zeta]-[\alpha,\alpha][\beta,\zeta] \\ \left\{\begin{matrix}[\beta,\alpha][\alpha,\xi]\\-[\alpha,\alpha][\beta,\xi]\end{matrix}\right\} & \left\{\begin{matrix}[\beta,\alpha][\alpha,\eta]\\-[\alpha,\alpha][\beta,\eta]\end{matrix}\right\} & \left\{\begin{matrix}[\beta,\alpha][\alpha,\zeta]\\-[\alpha,\alpha][\beta,\zeta]\end{matrix}\right\} & 0 \end{vmatrix} = 0 \qquad (114).$$

This can be written

$$\begin{aligned}&[\beta, \alpha]\begin{vmatrix} [\xi,\xi] & [\xi,\eta] & [\xi,\zeta] & [\alpha,\xi] \\ [\eta,\xi] & [\eta,\eta] & [\eta,\zeta] & [\alpha,\eta] \\ [\zeta,\xi] & [\zeta,\eta] & [\zeta,\zeta] & [\alpha,\zeta] \\ \left\{\begin{matrix}[\beta,\alpha][\alpha,\xi]\\-[\alpha,\alpha][\beta,\xi]\end{matrix}\right\} & \left\{\begin{matrix}[\beta,\alpha][\alpha,\eta]\\-[\alpha,\alpha][\beta,\eta]\end{matrix}\right\} & \left\{\begin{matrix}[\beta,\alpha][\alpha,\zeta]\\-[\alpha,\alpha][\beta,\zeta]\end{matrix}\right\} & \left\{\begin{matrix}[\beta,\alpha][\alpha,\alpha]\\-[\alpha,\alpha][\beta,\alpha]\end{matrix}\right\} \end{vmatrix}\\ &-[\alpha, \alpha]\begin{vmatrix} [\xi,\xi] & [\xi,\eta] & [\xi,\zeta] & [\beta,\xi] \\ [\eta,\xi] & [\eta,\eta] & [\eta,\zeta] & [\beta,\eta] \\ [\zeta,\xi] & [\zeta,\eta] & [\zeta,\zeta] & [\beta,\zeta] \\ \left\{\begin{matrix}[\beta,\alpha][\alpha,\xi]\\-[\alpha,\alpha][\beta,\xi]\end{matrix}\right\} & \left\{\begin{matrix}[\beta,\alpha][\alpha,\eta]\\-[\alpha,\alpha][\beta,\eta]\end{matrix}\right\} & \left\{\begin{matrix}[\beta,\alpha][\alpha,\zeta]\\-[\alpha,\alpha][\beta,\zeta]\end{matrix}\right\} & \left\{\begin{matrix}[\beta,\alpha][\alpha,\beta]\\-[\alpha,\alpha][\beta,\beta]\end{matrix}\right\} \end{vmatrix} = 0 \qquad (115).\end{aligned}$$

For the constituent in the fourth row and column of the last determinant vanishes by (76); and the constituent in the fourth row and column of the preceding determinant is identically zero.

Hence the condition becomes

$$[\beta, \alpha]^2 \frac{D\{[\xi], [\eta], [\zeta], [\alpha]\}}{D\{\xi, \eta, \zeta, \alpha\}}$$

$$- 2[\beta, \alpha][\alpha, \alpha] \frac{D\{[\xi], [\eta], [\zeta], [\alpha]\}}{D\{\xi, \eta, \zeta, \beta\}}$$

$$+ [\alpha, \alpha]^2 \frac{D\{[\xi], [\eta], [\zeta], [\beta]\}}{D\{\xi, \eta, \zeta, \beta\}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116).$$

And this is satisfied, because (30)–(32) hold.

Hence if (76) hold, the tangent plane to the conic node locus touches the tangent cone at the conic node.

Therefore the conic node locus is also an envelope.

(B.) Conversely, if the tangent plane to the conic node locus always touch the tangent cone at the conic node, then the condition

$$[\alpha, \alpha][\beta, \beta] - [\alpha, \beta]^2 = 0$$

is satisfied at every point of the conic node locus.

To determine the position of the tangent plane to the conic node locus, it is sufficient to eliminate $\delta\alpha$, $\delta\beta$ from any three of the equations (16), (17), (18), (28), (29), and then to use the relations (27).

Suppose that the values of $\delta\xi$, $\delta\eta$, $\delta\zeta$, which satisfy (16), (17), (18), (28), (29), are

$$\left.\begin{array}{l} \delta\xi = \lambda_1\,\delta\alpha + \lambda_2\,\delta\beta \\ \delta\eta = \mu_1\,\delta\alpha + \mu_2\,\delta\beta \\ \delta\zeta = \nu_1\,\delta\alpha + \nu_2\,\delta\beta \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117).$$

Then the tangent plane to the conic node locus is

$$(X - \xi)(\mu_1\nu_2 - \mu_2\nu_2) + (Y - \eta)(\nu_1\lambda_2 - \nu_2\lambda_1) + (Z - \zeta)(\lambda_1\mu_2 - \lambda_2\mu_1) = 0 \quad . \quad (118).$$

The condition that this may touch the tangent cone (33) is

$$\begin{vmatrix} [\xi, \xi] & [\eta, \xi] & [\zeta, \xi] & \mu_1\nu_2 - \mu_2\nu_1 \\ [\xi, \eta] & [\eta, \eta] & [\zeta, \eta] & \nu_1\lambda_2 - \nu_2\lambda_1 \\ [\xi, \zeta] & [\eta, \zeta] & [\zeta, \zeta] & \lambda_1\mu_2 - \lambda_2\mu_1 \\ \mu_1\nu_2 - \mu_2\nu_1 & \nu_1\lambda_2 - \nu_2\lambda_1 & \lambda_1\mu_2 - \lambda_2\mu_1 & 0 \end{vmatrix} = 0 . \qquad (119).$$

It will now be proved that the same condition can be obtained by substituting the values of $(\delta\xi)$, $(\delta\eta)$, $(\delta\zeta)$ from (117) in

$$[\xi, \xi](\delta\xi)^2 + [\eta, \eta](\delta\eta)^2 + [\zeta, \zeta](\delta\zeta)^2 + 2[\eta, \zeta](\delta\eta)(\delta\zeta) + 2[\zeta, \xi](\delta\zeta)(\delta\xi) + 2[\xi, \eta](\delta\xi)(\delta\eta) = 0 \quad . \quad . \quad (120),$$

and then making the roots of the resulting quadratic in $\delta\alpha/\delta\beta$ equal.

For making the substitution, the result is

$$(\delta\alpha)^2 \{[\xi, \xi]\lambda_1^2 + [\eta, \eta]\mu_1^2 + [\zeta, \zeta]\nu_1^2 + 2[\eta, \zeta]\mu_1\nu_1 + 2[\zeta, \xi]\nu_1\lambda_1 + 2[\xi, \eta]\lambda_1\mu_1\}$$

$$+ 2(\delta\alpha)(\delta\beta)\left[\begin{array}{c} [\xi, \xi]\lambda_1\lambda_2 + [\eta, \eta]\mu_1\mu_2 + [\zeta, \zeta]\nu_1\nu_2 \\ + [\eta, \zeta](\mu_1\nu_2 + \mu_2\nu_1) + [\zeta, \xi](\nu_1\lambda_2 + \nu_2\lambda_1) + [\xi, \eta](\lambda_1\mu_2 + \lambda_2\mu_1) \end{array}\right]$$

$$+ (\delta\beta)^2 \{[\xi, \xi]\lambda_2^2 + [\eta, \eta]\mu_2^2 + [\zeta, \zeta]\nu_2^2 + 2[\eta, \zeta]\mu_2\nu_2 + 2[\zeta, \xi]\nu_2\lambda_2 + 2[\xi, \eta]\lambda_2\mu_2\} = 0.$$

Now putting

$$\begin{aligned} L_1 &= [\xi, \xi]\lambda_1 + [\xi, \eta]\mu_1 + [\xi, \zeta]\nu_1, \\ M_1 &= [\eta, \xi]\lambda_1 + [\eta, \eta]\mu_1 + [\eta, \zeta]\nu_1, \\ N_1 &= [\zeta, \xi]\lambda_1 + [\zeta, \eta]\mu_1 + [\zeta, \zeta]\nu_1, \end{aligned}$$

and similar expressions for L_2, M_2, N_2, the condition for equal roots can be written

$$\begin{vmatrix} \lambda_1 L_1 + \mu_1 M_1 + \nu_1 N_1 & \lambda_1 L_2 + \mu_1 M_2 + \nu_1 N_2 \\ \lambda_2 L_1 + \mu_2 M_1 + \nu_2 N_1 & \lambda_2 L_2 + \mu_2 M_2 + \nu_2 N_2 \end{vmatrix} = 0.$$

It remains to show that this will be satisfied if (119) be satisfied.

Now

$$\begin{vmatrix} [\xi, \xi] & [\eta, \xi] & [\zeta, \xi] & \mu_1\nu_2 - \mu_2\nu_1 \\ [\xi, \eta] & [\eta, \eta] & [\zeta, \eta] & \nu_1\lambda_2 - \nu_2\lambda_1 \\ [\xi, \zeta] & [\eta, \zeta] & [\zeta, \zeta] & \lambda_1\mu_2 - \lambda_2\mu_1 \\ \mu_1\nu_2 - \mu_2\nu_1 & \nu_1\lambda_2 - \nu_2\lambda_1 & \lambda_1\mu_2 - \lambda_2\mu_1 & 0 \end{vmatrix} \times \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 & 0 \\ \lambda_2 & \mu_2 & \nu_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}^2$$

$$= \begin{vmatrix} L_1 & M_1 & N_1 & 0 \\ L_2 & M_2 & N_2 & 0 \\ [\zeta, \xi] & [\zeta, \eta] & [\zeta, \zeta] & \lambda_1\mu_2 - \lambda_2\mu_1 \\ \mu_1\nu_2 - \mu_2\nu_1 & \nu_1\lambda_2 - \nu_2\lambda_1 & \lambda_1\mu_2 - \lambda_2\mu_1 & 0 \end{vmatrix} \times \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 & 0 \\ \lambda_2 & \mu_2 & \nu_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \lambda_1 L_1 + \mu_1 M_1 + \nu_1 N_1 & \lambda_1 L_2 + \mu_1 M_2 + \nu_1 N_2 & N_1 & 0 \\ \lambda_2 L_1 + \mu_2 M_1 + \nu_2 N_1 & \lambda_2 L_2 + \mu_2 M_2 + \nu_2 N_2 & N_2 & 0 \\ N_1 & N_2 & [\zeta, \zeta] & \lambda_1\mu_2 - \lambda_2\mu_1 \\ 0 & 0 & \lambda_1\mu_2 - \lambda_2\mu_1 & 0 \end{vmatrix}$$

$$= -(\lambda_1\mu_2 - \lambda_2\mu_1)^2 \begin{vmatrix} \lambda_1 L_1 + \mu_1 M_1 + \nu_1 N_1 & \lambda_1 L_2 + \mu_1 M_2 + \nu_1 N_2 \\ \lambda_2 L_1 + \mu_2 M_1 + \nu_2 N_1 & \lambda_2 L_2 + \mu_2 M_2 + \nu_2 N_2 \end{vmatrix}$$

Hence dividing out by $(\lambda_1\mu_2 - \lambda_2\mu_1)^2$, it follows that the left-hand side of equation (119) is equal to

$$-\begin{vmatrix} \lambda_1 L_1 + \mu_1 M_1 + \nu_1 N_1 & \lambda_1 L_2 + \mu_1 M_2 + \nu_1 N_2 \\ \lambda_2 L_1 + \mu_2 M_1 + \nu_2 N_1 & \lambda_2 L_2 + \mu_2 M_2 + \nu_2 N_2 \end{vmatrix}.$$

Hence if the left-hand side of equation (119) vanish, so also does

$$\begin{vmatrix} \lambda_1 L_1 + \mu_1 M_1 + \nu_1 N_1 & \lambda_1 L_2 + \mu_1 M_2 + \nu_1 N_2 \\ \lambda_2 L_1 + \mu_2 M_1 + \nu_2 N_1 & \lambda_2 L_2 + \mu_2 M_2 + \nu_2 N_2 \end{vmatrix},$$

which was to be shown.

Hence either method of proceeding will lead to the condition that the tangent plane to the conic node locus should touch the tangent cone at the conic node.

The second method being simpler in this case will now be adopted.

Multiply (16) by $\delta\xi$, (17) by $\delta\eta$, (18) by $\delta\zeta$, (28) by $-(\delta\alpha)$, (29) by $-\delta\beta$, and add.

Hence

$$[\xi,\xi](\delta\xi)^2 + [\eta,\eta](\delta\eta)^2 + [\zeta,\zeta](\delta\zeta)^2 + 2[\eta,\zeta](\delta\eta)(\delta\zeta) + 2[\zeta,\xi](\delta\zeta)\delta\xi) + 2[\xi,\eta](\delta\xi)(\delta\eta)$$
$$= [\alpha,\alpha](\delta\alpha)^2 + 2[\alpha,\beta](\delta\alpha)(\delta\beta) + [\beta,\beta](\delta\beta)^2.$$

Hence the result of substituting the values of $\delta\xi$, $\delta\eta$, $\delta\zeta$, which satisfy (16), (17), (18), (28), (29) in (120) is

$$[\alpha,\alpha](\delta\alpha)^2 + 2[\alpha,\beta](\delta\alpha)(\delta\beta) + [\beta,\beta](\delta\beta)^2 = 0.$$

Forming the condition that the roots of this quadratic in $\delta\alpha/\delta\beta$ should be equal, it follows that

$$[\alpha,\alpha][\beta,\beta] - [\alpha,\beta]^2 = 0.$$

It will be proved in Art. 27 (see the equations (196)) that the common tangent line to the conic node and the conic node locus is in this case given by the equations

$$[\alpha,\xi](X-\xi) + [\alpha,\eta](Y-\eta) + [\alpha,\zeta](Z-\zeta) = 0,$$
$$[\beta,\xi](X-\xi) + [\beta,\eta](Y-\eta) + [\beta,\zeta](Z-\zeta) = 0.$$

Hence the common tangent line to the conic node and the conic node locus lies on the tangent planes to the surfaces $Df/D\alpha = 0$, $Df/D\beta = 0$; and it lies obviously on the tangent cone to the surface $f = 0$.

(C.) In this case equations (108) and (109) hold.

Also differentiating (109)

$$\frac{\partial^2\Delta}{\partial x^2} = \frac{\partial^2 R}{\partial x^2} f_1 f_2 + 2R\left[f_2 \frac{Df_1}{Dx} + f_1 \frac{Df_2}{Dx}\right]$$

$$+ R \left[\begin{array}{l} f_2\left(\frac{D^2 f_1}{Dx^2} + \frac{D^2 f_1}{DxDa_1}\frac{\partial a_1}{\partial x} + \frac{D^2 f_1}{DxDb_1}\frac{\partial b_1}{\partial x}\right) \\ + 2\frac{Df_1}{Dx}\frac{Df_2}{Dx} \\ + f_1\left(\frac{D^2 f_2}{Dx^2} + \frac{D^2 f_2}{DxDa_2}\frac{\partial a_2}{\partial x} + \frac{D^2 f_2}{DxDb_2}\frac{\partial b_2}{\partial x}\right) \end{array}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (121).$$

Substituting the values of $\partial a_1/\partial x$, $\partial b_1/\partial x$ from (52), there is in $\partial^2\Delta/\partial x^2$ the term

$$R f_2 \begin{vmatrix} [x, x] & [x, a_1] & [x, b_1] \\ [x, a_1] & [a_1, a_1] & [a_1, b_1] \\ [x, b_1] & [a_1, b_1] & [b_1, b_1] \end{vmatrix} \Bigg/ \begin{vmatrix} [a_1, a_1] & [a_1, b_1] \\ [a_1, b_1] & [b_1, b_1] \end{vmatrix}$$

which requires examination when $x = \xi$, $y = \eta$, $z = \zeta$, the coordinates of a point on the conic node locus.

Now in this case a_1, b_1 are roots of $Df_1/Da_1 = 0$, $Df_1/Db_1 = 0$.

Hence

$$[\alpha, \xi](\delta\xi) + [\alpha, \eta](\delta\eta) + [\alpha, \zeta](\delta\zeta) + [\alpha, \alpha](\delta\alpha) + [\alpha, \beta](\delta\beta)$$
$$+ \text{ terms of the second and higher orders in } \delta\xi, \delta\eta, \delta\zeta, \delta\alpha, \delta\beta = 0 \quad . \quad (122)$$

$$[\beta, \xi](\delta\xi) + [\beta, \eta](\delta\eta) + [\beta, \zeta](\delta\zeta) + [\beta, \alpha](\delta\alpha) + [\beta, \beta](\delta\beta)$$
$$+ \text{ terms of the second and higher orders in } \delta\xi, \delta\eta, \delta\zeta, \delta\alpha, \delta\beta = 0 \quad . \quad (123).$$

Multiply (122) by $[\alpha, \beta,]$, (123) by $[\alpha, \alpha]$ and subtract, the terms of the first order in $\delta\alpha$, $\delta\beta$ disappear, and the equation obtained is of the form :—

$$(\text{terms of the first order in } \delta\xi, \delta\eta, \delta\zeta)$$
$$+ (\text{terms of the second and higher orders in } \delta\xi, \delta\eta, \delta\zeta, \delta\alpha, \delta\beta) = 0.$$

Hence if $\delta\xi$, $\delta\eta$, $\delta\zeta$ are of the order of the infinitely small quantity ϵ; then $\delta\alpha$, $\delta\beta$ are of the order of $\epsilon^{\frac{1}{2}}$.

Hence the principal terms in (122) and (123) are $[\alpha, \alpha](\delta\alpha) + [\alpha, \beta]\,\delta\beta$ and $[\beta, \alpha](\delta\alpha) + [\beta, \beta](\delta\beta)$ respectively.

Moreover by (122) and (123), although $\delta\alpha$, $\delta\beta$ are of the order $\epsilon^{\frac{1}{2}}$, yet $[\alpha, \alpha]\,\delta\alpha + [\alpha, \beta]\,\delta\beta$ being ultimately equal to $-\{[\alpha, \xi](\delta\xi) + [\alpha, \eta]\,\delta\eta + [\alpha, \zeta]\,\delta\zeta\}$ is of the order ϵ.

Similarly $[\beta, \alpha]\,\delta\alpha + [\beta, \beta]\,\delta\beta$ is of the order ϵ.

Next, when $x = \xi$, $y = \eta$, $z = \zeta$, $a_2 = \alpha + \delta\alpha'$, $b_2 = \beta + \delta\beta'$, $f(x, y, z, a_2, b_2)$ becomes

$$
\begin{aligned}
&f(\xi, \eta, \zeta, \alpha, \beta) \\
&+ [\xi](\delta\xi) + [\eta](\delta\eta) + [\zeta](\delta\zeta) + [\alpha](\delta\alpha') + [\beta](\delta\beta') \\
&+ \tfrac{1}{2}\left[\begin{aligned}
&[\xi, \xi](\delta\xi)^2 + [\eta, \eta](\delta\eta)^2 + [\zeta, \zeta](\delta\zeta)^2 \\
&+ 2[\eta, \zeta](\delta\eta)(\delta\zeta) + 2[\zeta, \xi](\delta\zeta)(\delta\xi) + 2[\xi, \eta](\delta\xi)(\delta\eta) \\
&+ 2(\delta\alpha')\{[\xi, \alpha](\delta\xi) + [\eta, \alpha](\delta\eta) + [\zeta, \alpha](\delta\zeta)\} \\
&+ 2(\delta\beta')\{[\xi, \beta](\delta\xi) + [\eta, \beta](\delta\eta) + [\zeta, \beta](\delta\zeta)\} \\
&+ [\alpha, \alpha](\delta\alpha')^2 + 2[\alpha, \beta](\delta\alpha')(\delta\beta') + [\beta, \beta](\delta\beta')^2
\end{aligned}\right] \\
&+ \text{terms of the third and higher orders in } \delta\xi, \delta\eta, \delta\zeta, \delta\alpha', \delta\beta' \\
= \;& \quad (\text{terms of the second order in } \delta\xi, \delta\eta, \delta\zeta) \\
&+ (\delta\alpha')(\text{terms of the first order in } \delta\xi, \delta\eta, \delta\zeta) \\
&+ (\delta\beta')(\text{terms of the first order in } \delta\xi, \delta\eta, \delta\zeta) \\
&+ \frac{1}{2[\alpha, \alpha]}\{[\alpha, \alpha](\delta\alpha') + [\alpha, \beta]\delta\beta'\}^2 \\
&+ \text{terms of the third and higher orders in } \delta\xi, \delta\eta, \delta\zeta, \delta\alpha', \delta\beta'.
\end{aligned}
$$

Now, the terms of the second order in $\delta\xi$, $\delta\eta$, $\delta\zeta$ are of order ϵ^2.

The terms containing $\delta\alpha'$ or $\delta\beta'$, multiplied by terms of the first order in $\delta\xi$, $\delta\eta$, $\delta\zeta$, are of order $\epsilon^{3/2}$.

The terms $\frac{1}{2[\alpha, \alpha]}\{[\alpha, \alpha](\delta\alpha') + [\alpha, \beta]\delta\beta'\}^2$ are of order ϵ^2, since $[\alpha, \alpha](\delta\alpha') + [\alpha, \beta](\delta\beta')$ is of order ϵ, by the same argument as the one which was applied to show that $[\alpha, \alpha](\delta\alpha) + [\alpha, \beta](\delta\beta)$ was of order ϵ.

The most important terms of the third and higher orders in $\delta\xi$, $\delta\eta$, $\delta\zeta$, $\delta\alpha'$, $\delta\beta'$ are of order $\epsilon^{3/2}$.

Hence $f(x, y, z, a_2, b_2)$ is of order $\epsilon^{3/2}$, when $x = \xi$, $y = \eta$, $z = \zeta$, the coordinates of a point on the conic node locus.

Further

$$\begin{vmatrix} [a_1, a_1] & [a_1, b_1] \\ [a_1, b_1] & [b_1, b_1] \end{vmatrix}$$

becomes

$$
\begin{aligned}
&\begin{vmatrix} [\alpha, \alpha] & [\alpha, \beta] \\ [\alpha, \beta] & [\beta, \beta] \end{vmatrix} \\
&+ \left\{(\delta\xi)\frac{D}{D\xi} + (\delta\eta)\frac{D}{D\eta} + (\delta\zeta)\frac{D}{D\zeta} + (\delta\alpha)\frac{D}{D\alpha} + (\delta\beta)\frac{D}{D\beta}\right\}\begin{vmatrix} [\alpha, \alpha] & [\alpha, \beta] \\ [\alpha, \beta] & [\beta, \beta] \end{vmatrix} \\
&+ \text{terms of the second and higher orders in } \delta\xi, \delta\eta, \delta\zeta, \delta\alpha, \delta\beta.
\end{aligned}
$$

Hence, when $x = \xi$, $y = \eta$, $z = \zeta$,

$$\begin{vmatrix} [a_1, a_1] & [a_1, b_1] \\ [a_1, b_1] & [b_1, b_1] \end{vmatrix}$$

is of order $\epsilon^{1/2}$.

Hence

$$f_2 \Big/ \begin{vmatrix} [a_1, a_1] & [a_1, b_1] \\ [a_1, b_1] & [b_1, b_1] \end{vmatrix}$$

is of order ϵ, and therefore vanishes at points on the conic node locus.

Similarly it can be shown that the term

$$f_1 \Big/ \begin{vmatrix} [a_2, a_2] & [a_2, b_2] \\ [a_2, b_2] & [b_2, b_2] \end{vmatrix}$$

vanishes at points on the conic node locus.

Therefore Δ, $\partial\Delta/\partial x$, $\partial^2\Delta/\partial x^2$ all vanish on the conic node locus.

Therefore Δ contains C^3 as a factor.

Example 9.—*Conic Node Locus which is also an Envelope.*

Let the surfaces be

$$\alpha (x - a)^3 + 3\beta (y - b)^2 + 3\gamma (x - a) z + \delta z^2 = 0,$$

where α, β, γ, δ are fixed constants; a, b the arbitrary parameters.

(A.) *The Discriminant.*

It is the same as that of the equation

$$\alpha X^3 + 3\beta Y^2 Z + 3\gamma z X Z^2 + \delta z^2 Z^3 = 0.$$

Therefore

$$S = \alpha\beta^2\gamma z,$$
$$T = 4\alpha^2\beta^3 \delta z^2.$$

Therefore

$$\Delta = 16\alpha^3\beta^6 z^3 (\alpha\delta^2 z + 4\gamma^3).$$

(B.) *The Conic Node Locus, which is also an Envelope, is* $z = 0$.

To prove this, transform the equation by means of $x = a + X$, $y = b + Y$, $z = Z$. It becomes

$$\alpha X^3 + 3\beta Y^2 + 3\gamma XZ + \delta Z^2 = 0.$$

Hence the new origin is a conic node, and one of the tangent planes of the conic node is $Z = 0$.

Hence the conic node locus is $z = 0$, and it is also an envelope.

(C.) *The Locus* $\alpha\delta^2 z + 4\gamma^3 = 0$ *is an Ordinary Envelope.*

This can be proved by finding the tangent planes parallel to $z = 0$.

Hence it is necessary to satisfy at the same time

$$\begin{aligned} &\alpha (x - a)^3 + 3\beta (y - b)^2 + 3\gamma (x - a) z + \delta z^2 = 0, \\ &\alpha (x - a)^2 + \gamma z = 0, \\ &y - b = 0, \\ &3\gamma (x - a) + 2 \delta z \neq 0 \quad . \quad . \quad . \quad (124). \end{aligned}$$

Hence

$$y = b,$$

therefore

$$\begin{aligned} &\alpha (x - a)^3 + 3\gamma (x - a) z + \delta z^2 = 0, \\ &\alpha (x - a)^2 + \gamma z = 0. \end{aligned}$$

Hence

$$2\gamma (x - a) z + \delta z^2 = 0.$$

The solution $z = 0$ of the last equation makes $x = a$, and does not satisfy (124).

Hence it is necessary to take

$$2\gamma (x - a) + \delta z = 0.$$

This gives

$$\alpha\delta^2 z + 4\gamma^3 = 0.$$

Hence, when

$$x = a + 2\gamma^2/\alpha\delta, \quad y = b, \quad z = - 4\gamma^3/\alpha\delta^2,$$

the tangent plane is parallel to the plane $z = 0$. It touches all the surfaces of the system.

Hence

$$z = - 4\gamma^3/\alpha\delta^2$$

is an ordinary envelope.

Art. 15.—*To prove that if the Edge of the Biplanar Node always touch the Biplanar Node Locus, then* Δ *contains* B^4 *as a factor.*

(A.) It will be shown that when the condition (76) holds in the case of a biplanar node locus, then the edge of the biplanar node always touches the biplanar node locus.

The equation of the biplanes is given by (33).

Now, if the left-hand side of (33) break up into two linear factors, then the two planes, whose equations are given by equating the two linear factors to zero, will intersect in the straight line whose equations are given by any two of the three equations :—

$$[\xi, \xi] (X - \xi) + [\xi, \eta] (Y - \eta) + [\xi, \zeta] (Z - \zeta) = 0,$$
$$[\eta, \xi] (X - \xi) + [\eta, \eta] (Y - \eta) + [\eta, \zeta] (Z - \zeta) = 0,$$
$$[\zeta, \xi] (X - \xi) + [\zeta, \eta] (Y - \eta) + [\zeta, \zeta] (Z - \zeta) = 0 \quad . \quad . \quad . \quad (125).$$

To find the tangent plane to the binode locus, proceed thus :—

The condition (41) gives

$$[\xi, \xi] \{[\alpha, \alpha] [\beta, \beta] - [\alpha, \beta]^2\}$$
$$- [\xi, \alpha]^2 [\beta, \beta] + 2 [\xi, \alpha] [\xi, \beta] [\alpha, \beta] - [\xi, \beta]^2 [\alpha, \alpha] = 0,$$

which, by means of (76), can be written

$$[\xi, \alpha]^2 [\alpha, \beta]^2 - 2 [\xi, \alpha] [\xi, \beta] [\alpha, \beta] [\alpha, \alpha] + [\xi, \beta]^2 [\alpha, \alpha]^2 = 0.$$

Therefore

$$[\xi, \alpha] [\alpha, \beta] - [\xi, \beta] [\alpha, \alpha] = 0.$$

Similarly

$$[\eta, \alpha] [\alpha, \beta] - [\eta, \beta] [\alpha, \alpha] = 0,$$
$$[\zeta, \alpha] [\alpha, \beta] - [\zeta, \beta] [\alpha, \alpha] = 0.$$

Hence

$$\frac{[\xi, \alpha]}{[\xi, \beta]} = \frac{[\eta, \alpha]}{[\eta, \beta]} = \frac{[\zeta, \alpha]}{[\zeta, \beta]} = \frac{[\alpha, \alpha]}{[\alpha, \beta]} = \frac{[\beta, \alpha]}{[\beta, \beta]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126).$$

Now, multiplying (16) by $[\eta, \alpha]$, (17) by $[\xi, \alpha]$ and subtracting

$$[\eta, \alpha] \{[\xi, \xi] (\delta\xi) + [\xi, \eta] (\delta\eta) + [\xi, \zeta] (\delta\zeta)\}$$
$$- [\xi, \alpha] \{[\eta, \xi] (\delta\xi) + [\eta, \eta] (\delta\eta) + [\eta, \zeta] (\delta\zeta)\}$$
$$+ (\delta\beta) \{[\eta, \alpha] [\xi, \beta] - [\eta, \beta] [\xi, \alpha]\} = 0.$$

Now, by (126), the coefficient of $\delta\beta$ vanishes, and the equation of the tangent plane to the binode locus is

$$[\eta, \alpha] \{[\xi, \xi] (X - \xi) + [\xi, \eta] (Y - \eta) + [\xi, \zeta] (Z - \zeta)\}$$
$$- [\xi, \alpha] \{[\eta, \xi] (X - \xi) + [\eta, \eta] (Y - \eta) + [\eta, \zeta] (Z - \zeta)\} = 0 \quad . \quad (127).$$

Hence the tangent plane to the binode locus passes through the intersection of two of the planes (125), and, therefore, through the edge of the binode.

Hence the edge of the binode always touches the binode locus.

It may be noticed, further, with respect to the edge of the binode, since equations (28) and (29) depend on (16) and (17), that since it is the intersection of the planes (125) it lies also on the planes

$$[\alpha, \xi](X - \xi) + [\alpha, \eta](Y - \eta) + [\alpha, \zeta](Z - \zeta) = 0,$$
$$[\beta, \xi](X - \xi) + [\beta, \eta](Y - \eta) + [\beta, \zeta](Z - \zeta) = 0.$$

(These planes, it may be noted, coincide in this case by (126).)

Hence the edge lies on the tangent planes to the surfaces $Df/D\alpha = 0$, $Df/D\beta = 0$, and it is obviously a tangent to the surface $f = 0$.

(B.) Conversely, if the edge of the binode always touches the binode locus, then the condition (76) holds at every point of the binode locus.

The equation of the edge is given by any two of the three equations (125).

Hence, if the edge is a tangent line to the locus of binodes, the equations (125) will be satisfied by putting

$$X = \xi + \delta\xi, \quad Y = \eta + \delta\eta, \quad Z = \zeta + \delta\zeta,$$

the coordinates of a binode near to ξ, η, ζ, which lies on the edge of the first binode and infinitely near to it.

Hence

$$\left.\begin{aligned}[\xi, \xi](\delta\xi) + [\xi, \eta](\delta\eta) + [\xi, \zeta](\delta\zeta) = 0,\\ [\eta, \xi](\delta\xi) + [\eta, \eta](\delta\eta) + [\eta, \zeta](\delta\zeta) = 0,\\ [\zeta, \xi](\delta\xi) + [\zeta, \eta](\delta\eta) + [\zeta, \zeta](\delta\zeta) = 0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (128).$$

But equations (16), (17), (18), (28), (29), also hold.

Hence, by (128) it follows that (16), (17), (18) become

$$[\xi, \alpha](\delta\alpha) + [\xi, \beta](\delta\beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129),$$

$$[\eta, \alpha](\delta\alpha) + [\eta, \beta](\delta\beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130),$$

$$[\zeta, \alpha](\delta\alpha) + [\zeta, \beta](\delta\beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

Hence

$$[\xi, \alpha]/[\xi, \beta] = [\eta, \alpha]/[\eta, \beta] = [\zeta, \alpha]/[\zeta, \beta] \quad . \quad . \quad . \quad . \quad . \quad (132).$$

Now only two of the five equations (16), (17), (18), (28), (29) are independent.

Suppose that (16), (17) are independent.

Then, since (28), (29) depend on these, relations exist of the form

$$\begin{aligned}
[\xi, \alpha] &= \lambda[\xi, \xi] + \mu[\xi, \eta], & [\xi, \beta] &= \rho[\xi, \xi] + \sigma[\xi, \eta],\\
[\eta, \alpha] &= \lambda[\xi, \eta] + \mu[\eta, \eta], & [\eta, \beta] &= \rho[\xi, \eta] + \sigma[\eta, \eta],\\
[\zeta, \alpha] &= \lambda[\xi, \zeta] + \mu[\zeta, \eta], & [\zeta, \beta] &= \rho[\xi, \zeta] + \sigma[\eta, \zeta],\\
[\alpha, \alpha] &= \lambda[\xi, \alpha] + \mu[\alpha, \eta], & [\alpha, \beta] &= \rho[\xi, \alpha] + \sigma[\eta, \alpha],\\
[\alpha, \beta] &= \lambda[\xi, \beta] + \mu[\beta, \eta] & [\beta, \beta] &= \rho[\xi, \beta] + \sigma[\eta, \beta].
\end{aligned}$$

Hence, by (132),

$$\frac{\lambda[\xi,\xi]+\mu[\xi,\eta]}{\rho[\xi,\xi]+\sigma[\xi,\eta]}=\frac{\lambda[\xi,\eta]+\mu[\eta,\eta]}{\rho[\xi,\eta]+\sigma[\eta,\eta]}=\frac{\lambda[\xi,\zeta]+\mu[\eta,\zeta]}{\rho[\xi,\zeta]+\sigma[\eta,\zeta]}.$$

Therefore

$$[\lambda\sigma-\mu\rho]\{[\xi,\xi][\eta,\eta]-[\xi,\eta]^2\}=0,$$

and

$$[\lambda\sigma-\mu\rho]\{[\xi,\xi][\eta,\zeta]-[\xi,\eta][\xi,\zeta]\}=0.$$

Hence, unless $\lambda\sigma-\mu\rho=0$, it is necessary to have both

$$[\xi,\xi][\eta,\eta]-[\xi,\eta]^2=0,$$

and

$$[\xi,\xi][\eta,\zeta]-[\xi,\eta][\xi,\zeta]=0.$$

Hence

$$[\xi,\xi]/[\xi,\eta]=[\xi,\eta]/[\eta,\eta]=[\xi,\zeta]/[\eta,\zeta].$$

But if these results hold, the two equations taken to determine the edge of the binode would be the same, and would not determine it. Supposing then that those two equations have been selected, which are independent, this alternative cannot hold, and therefore

$$\lambda\sigma-\mu\rho=0.$$

Therefore

$$\frac{[\xi,\alpha]}{[\xi,\beta]}=\frac{[\eta,\alpha]}{[\eta,\beta]}=\frac{[\zeta,\alpha]}{[\zeta,\beta]}=\frac{[\alpha,\alpha]}{[\alpha,\beta]}=\frac{[\alpha,\beta]}{[\beta,\beta]}=\frac{\lambda}{\rho}.$$

Hence the equations (126) are satisfied, and in particular

$$[\alpha,\alpha][\beta,\beta]-[\alpha,\beta]^2=0.$$

Hence, if the edge of the binode always touch the binode locus, the condition (76) holds.

(C.) In this case Δ is given by (108).

Let ξ, η, ζ be any point on the binode locus.

Then when $x=\xi$, $y=\eta$, $z=\zeta$,

$a_1=a_2=a$ of surface having a binode at ξ, η, ζ,

$b_1=b_2=b$ of surface having a binode at ξ, η, ζ.

$Df_1/Dx=0$, $Df_1/Dy=0$, $Df_1/Dz=0$, when $x=\xi$, $y=\eta$, $z=\zeta$.

$Df_2/Dx=Df_1/Dx$, when $x=\xi$, $y=\eta$, $z=\zeta$.

Therefore, $Df_2/Dx=0$, and similarly $Df_2/Dy=0$, $Df_2/Dz=0$, when $x=\xi$, $y=\eta$, $z=\zeta$.

Now if each of the differential coefficients of Δ be formed up to the third order, then every term in the result will contain as a factor one of the following quantities :—

$$f_1 \text{ or } f_2 \text{ or a first differential coefficient of } f_1 \text{ or } f_2.$$

Hence, when $x = \xi$, $y = \eta$, $z = \zeta$, Δ and all its differential coefficients up to the third order vanish.

Hence, if $B = 0$ be the equation of the binode locus, such that the edge of the binode always touches the binode locus, Δ contains B^4 as a factor by Art. 1, Preliminary Theorem B.

Example 10.—*Locus of Biplanar Nodes such that the Edge of the Biplanar Node always touches the Biplanar Node Locus.*

Let the surfaces be

$$\alpha\,(x-a)^3 + \beta\,(y-b)^3 + 3\,[c\,(x-a) + e\,(y-b) + gz]^2 - h^2z^2 = 0,$$

where α, β, c, e, g, h are fixed constants ; a, b are the arbitrary parameters.

(A.) *The Discriminant.*

It is the same as that of the equation

$$\alpha X^3 + \beta Y^3 + 3Z\,(cX + eY + gzZ)^2 - h^2z^2Z^3 = 0.$$

Hence,

$$S = -\,\alpha\beta ce\,(g^2 + h^2)\,z^2,$$

$$T = \alpha^2\beta^2\,(3g^2 - h^2)^2\,z^4 + 4\alpha\beta g\,(\alpha e^3 + \beta c^3)\,(3h^2 - g^2)\,z^3 - 4h^2\,(\alpha e^3 - \beta c^3)^2\,z^2.$$

Therefore,

$$\Delta = z^4 \left[\begin{array}{l} \{(\alpha^2\beta^2)\,(3g^2 - h^2)^2\,z^2 + 4\alpha\beta g\,(\alpha e^3 + \beta c^3)\,(3h^2 - g^2)\,z - 4h^2\,(\alpha e^3 - \beta c^3)^2\}^2 \\ \qquad - 64\alpha^3\beta^3c^3e^3\,(g^2 + h^2)^3\,z^2 \end{array} \right]$$

In order to show the way in which the factor z^4 arises, the method in which the discriminant is formed will now be examined.

It may be obtained by eliminating X, Y, Z from

$$(cX + eY + gzZ)\,(cX + eY + 3gzZ) - h^2z^2Z^2 = 0 \quad . \quad . \quad . \quad (133),$$

$$\alpha X^2 + 2cZ\,(cX + eY + gzZ) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134),$$

$$\beta Y^2 + 2eZ\,(cX + eY + gzZ) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (135).$$

Hence,

$$Y = \pm\,X\,\sqrt{(e\alpha/c\beta)}.$$

Representing both values of Y by $Y = \lambda X$, it follows that

$$\alpha X^2 + 2c(c + e\lambda) XZ + 2cgzZ^2 = 0.$$

Therefore,

$$X/Z = -\frac{c}{\alpha}(c + e\lambda) \pm \frac{1}{\alpha}\{c^2(c + e\lambda)^2 - 2c\alpha gz\}^{1/2}.$$

Hence,

$$x - a = -\frac{c}{\alpha}(c + e\lambda) \pm \frac{1}{\alpha}\{c^2(c + e\lambda)^2 - 2c\alpha gz\}^{1/2} \quad . \quad . \quad . \quad . \quad (136),$$

$$y - b = \lambda\left\{-\frac{c}{\alpha}(c + e\lambda) \pm \frac{1}{\alpha}\{c^2(c + e\lambda)^2 - 2c\alpha gz\}^{1/2}\right\} \quad . \quad . \quad . \quad (137).$$

These give the values of a, b which, when substituted in the equation of the surfaces, give the discriminant.

The values of a, b corresponding to a point ξ, η, ζ on the binode locus, will now be found.

It will be shown presently that $z = 0$ is the binode locus.

Hence $\zeta = 0$, and therefore

$$\xi - a = -\frac{c}{\alpha}(c + e\lambda) \pm \frac{c}{\alpha}(c + e\lambda),$$

$$\eta - b = \lambda\left\{-\frac{c}{\alpha}(c + e\lambda) \pm \frac{c}{\alpha}(c + e\lambda)\right\}.$$

Hence, for *each* value of λ, one of the values of a is ξ, and one of the values of b is η.

Hence there are two sets of values of a, b satisfying $Df/Da = 0$, $Df/Db = 0$, which become equal when $x = \xi$, $y = \eta$, $z = 0$.

These two sets of values both give $a = \xi$, $b = \eta$.

It will now be shown that the substitution of each of these systems of values of a, b in f, will give rise to the factor z^2 in Δ.

Now

$$Y = \lambda X, \qquad X = \mu Z,$$

where

$$\lambda = \pm\sqrt{(e\alpha/c\beta)},$$

$$\mu = -\frac{c}{\alpha}(c + e\lambda) \pm \frac{1}{\alpha}\{c^2(c + e\lambda)^2 - 2c\alpha gz\}^{1/2}.$$

Substituting these in the left-hand side of (133), it becomes

$$[(c\mu + e\lambda\mu + gz)(c\mu + e\lambda\mu + 3gz) - h^2z^2]Z^2,$$

i.e.,

$$[\mu^2(c + e\lambda)^2 + 4g\mu(c + e\lambda)z + (3g^2 - h^2)z^2]Z^2.$$

Although, to find the discriminant in the usual way, it is necessary to substitute each set of values of λ, μ in this expression, and then to multiply the results together, it is possible to determine more readily which value of μ will lead to the factor z^2 by expanding μ in ascending powers of z.

Now

$$\mu = -\frac{c}{\alpha}(c+e\lambda) \pm \frac{c}{\alpha}(c+e\lambda)\left\{1 - \frac{\alpha g z}{c(c+e\lambda)^2} - \frac{\alpha^2 g^2 z^2}{2c^2(c+e\lambda)^4} - \ldots\right\}.$$

Taking the upper sign

$$\mu = -\frac{gz}{c+e\lambda} - \frac{\alpha g^2 z^2}{2c(c+e\lambda)^3} - \ldots$$

Substituting this value of μ in

$$[\mu^2(c+e\lambda)^2 + 4g\mu(c+e\lambda)z + (3g^2-h^2)z^2]Z^2,$$

the coefficient of z^2 in the bracket is $(-h^2)$, there being no lower power of z.

This being true for each value of λ, the factor z^4 is accounted for.

The other value of μ will lead to a factor, in which there is a term independent of z.

The elimination will now be completed.

It is necessary to substitute the values of λ and μ from

$$\lambda = \pm\sqrt{(e\alpha/c\beta)},$$

and

$$\alpha\mu^2 + 2c(c+e\lambda)\mu + 2cgz = 0,$$

in

$$\mu^2(c+e\lambda)^2 + 4g\mu(c+e\lambda)z + (3g^2-h^2)z^2.$$

Substituting first for μ^2, and multiplying by α, this becomes

$$\mu(c+e\lambda)[-2c(c+e\lambda)^2 + 4g\alpha z] + [-2cgz(c+e\lambda)^2 + \alpha z^2(3g^2-h^2)].$$

Substituting both values of μ in this, multiplying the results together, and multiplying by α, the result is

$$\begin{aligned}&2cgz(c+e\lambda)^2[-2c(c+e\lambda)^2 + 4g\alpha z]^2\\ &\quad - 2c(c+e\lambda)^2[-2c(c+e\lambda)^2 + 4g\alpha z][-2cgz(c+e\lambda)^2 + \alpha z^2(3g^2-h^2)]\\ &\quad + \alpha[-2cgz(c+e\lambda)^2 + \alpha z^2(3g^2-h^2)]^2.\end{aligned}$$

This reduces to

$$\begin{aligned}&-4c^2\alpha h^2(c+e\lambda)^4 z^2\\ &+4cg\alpha^2(3h^2-g^2)(c+e\lambda)^2 z^3\\ &+\alpha^3(3g^2-h^2)^2 z^4.\end{aligned}$$

Putting $\lambda^2 = e\alpha/c\beta$, and multiplying by β^2/α, this becomes

$$\alpha^2\beta^2(3g^2 - h^2)^2 z^4 + 4g\alpha\beta(3h^2 - g^2)(\alpha e^3 + \beta c^3) z^3 - 4h^2 z^2 (\alpha e^3 - \beta c^3)^2$$
$$- 32\alpha e^3 \beta c^3 h^2 z^2 + 8c^2 e\beta\lambda\,[g\alpha\beta z^3 (3h^2 - g^2) - 2h^2 z^2 (\alpha e^3 + \beta c^3)].$$

Substituting for λ its two values $\pm\sqrt{(e\alpha/c\beta)}$, multiplying the results together, and reducing, it becomes

$$z^4\left[\begin{array}{c}\{\alpha^2\beta^2(3g^2 - h^2)^2 z^2 + 4\alpha\beta g(\alpha e^3 + \beta c^3)(3h^2 - g^2) z - 4h^2(\alpha e^3 - \beta c^3)^2\}^2 \\ - 64\alpha^3\beta^3 c^3 e^3 (g^2 + h^2)^3 z^2\end{array}\right]$$

This is the same value as before for the discriminant.

(B.) *The Surface $z = 0$ is a Binode Locus such that the Edge of the Binode touches the Binode Locus.*

Transforming the equation by means of $x = a + \mathrm{X}$, $y = b + \mathrm{Y}$, $z = \mathrm{Z}$, it becomes

$$\alpha\mathrm{X}^3 + \beta\mathrm{Y}^3 + 3(c\mathrm{X} + e\mathrm{Y} + g\mathrm{Z})^2 - h^2\mathrm{Z}^2 = 0.$$

Hence the new origin is a binode.

Hence the binode locus is $z = 0$.

The biplanes are

$$3^{\frac{1}{2}}(c\mathrm{X} + e\mathrm{Y} + g\mathrm{Z}) - h\mathrm{Z} = 0.$$
$$3^{\frac{1}{2}}(c\mathrm{X} + e\mathrm{Y} + g\mathrm{Z}) + h\mathrm{Z} = 0.$$

The equations of the edge are therefore

$$c\mathrm{X} + e\mathrm{Y} = 0, \quad \mathrm{Z} = 0.$$

It lies therefore in the plane $\mathrm{Z} = 0$, *i.e.*, in the plane $z = 0$.

Hence it may be considered to touch the binode locus.

The condition (76) is satisfied at every point on the binode locus.

Hence the factor z^4 is accounted for.

(C.) *The Surface*

$$\{\alpha^2\beta^2(3g^2 - h^2)^2 z^2 + 4g\alpha\beta(\alpha e^3 + \beta c^3)(3h^2 - g^2) z - 4h^2(\alpha e^3 - \beta c^3)^2\}^2$$
$$- 64\,\alpha^3\beta^3 c^3 e^3 (g^2 + h^2)^3 z^2 = 0$$

is an ordinary Envelope.

This may be proved by finding the tangent planes parallel to the plane $z = 0$.

Hence it is necessary to satisfy at the same time

$$\alpha (x - a)^3 + \beta (y - b)^3 + 3 [c (x - a) + e (y - b) + gz]^2 - h^2 z^2 = 0 \quad (138),$$

$$\alpha (x - a)^2 + 2c [c (x - a) + e (y - b) + gz] = 0 \quad (139),$$

$$\beta (y - b)^2 + 2e [c (x - a) + e (y - b) + gz] = 0 \quad (140),$$

$$3g [c (x - a) + e (y - b) + gz] - h^2 z \neq 0 \quad (141),$$

or putting $x - a = X/Z$, $y - b = Y/Z$, the equations (138)-(140) become the same as (133), (134), (135).

Hence the result of the elimination will be the same as in the previous case. It is only necessary to show that (141) is satisfied.

Multiplying (139) by $(x - a)$, (140) by $(y - b)$ and subtracting from (138), it follows that

$$[c (x - a) + e (y - b) + gz] [c (x - a) + e (y - b) + 3gz] - h^2 z^2 = 0.$$

Therefore

$$[c (x - a) + e (y - b) + gz]^2 + 2gz [c (x - a) + e (y - b) + gz] - h^2 z^2 = 0.$$

Therefore

$$c (x - a) + e (y - b) + gz = - gz \pm z \sqrt{(g^2 + h^2)}.$$

Hence (141) is not satisfied unless $z = 0$.

Now $z = 0$ makes $c (x - a) + e (y - b) + gz = 0$.

Therefore $x = a$ by (139) and $y = b$ by (140).

This solution corresponds to the binode locus.

It may therefore be excluded.

Hence the factor of the discriminant under discussion corresponds to an envelope locus, touching all the surfaces; it consists of four planes parallel to $z = 0$, whose equations are independent of the arbitrary parameters.

SECTION IV. (Arts. 16–25).—CONSIDERATION OF CASES RESERVED FROM THE PREVIOUS SECTION. THE DEGREE OF $f(x, y, z, a, b)$ IN a, b IS NOW THE SECOND, AND THE EQUATIONS $Df/Da = 0$, $Df/Db = 0$ ARE INDETERMINATE EQUATIONS FOR THE PARAMETERS AT POINTS ON THE LOCUS OF ULTIMATE INTERSECTIONS.

It was supposed in the previous section that the degree of $f(x, y, z, a, b)$ in a, b was higher than the second; for if the degree were the second, and the analytical condition satisfied which expresses that at a point on the locus of ultimate intersections, two systems of values of the parameters, which satisfy $Df/Da = 0$, $Df/Db = 0$, become equal, then this analytical condition requires to be specially interpreted.

For now $Df/Da = 0$, $Df/Db = 0$ are two simple equations in a, b. Hence they are either satisfied by one value of a and one of b, or else are indeterminate. But since the condition

$$\frac{D^2f}{Da^2}\frac{D^2f}{Db^2} - \left(\frac{D^2f}{Da\,Db}\right)^2 = 0$$

holds, they are indeterminate.

In this case the discriminant cannot be formed as in the previous section.

There are not two coinciding systems of values of the parameters to consider. It is shown that there is one system which can be determined.

There is also the additional peculiarity that the rationalising factor introduced to make the discriminant of the proper order and weight always vanishes at a point on the locus of ultimate intersections. Hence, on account of it, the equation of an envelope or singular point locus may be expected to enter into the discriminant one or more times. As this number cannot be determined in a general way, it is better to express the equation of the system of surfaces as a quadric function of the parameters, and form the discriminant in the usual way.

Art. 16.—*The Discriminant and its Differential Coefficients as far as the third order.*

Let the equation of the system of surfaces be

$$ua^2 + 2Wab + vb^2 + 2Va + 2Ub + w = 0 \quad . \quad . \quad . \quad . \quad (142).$$

To find the discriminant, solve for a, b, the equations

$$ua + Wb + V = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (143).$$

$$Wa + vb + U = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144),$$

obtaining hence

$$\left.\begin{aligned} a &= \frac{WU - vV}{uv - W^2} \\ b &= \frac{WV - uU}{uv - W^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (145).$$

Now substitute these values of a, b in the left-hand side of (142).

The result is

$$\begin{vmatrix} u & W & V \\ W & v & U \\ V & U & w \end{vmatrix} \div \begin{vmatrix} u & W \\ W & v \end{vmatrix}.$$

The rationalising factor is $\begin{vmatrix} u & W \\ W & v \end{vmatrix}$.

Hence the discriminant

$$\Delta = \begin{vmatrix} u & W & V \\ W & v & U \\ V & U & w \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (146).$$

Therefore

$$\Delta_x = \begin{vmatrix} u_x & W_x & V_x \\ W & v & U \\ V & U & w \end{vmatrix} + \begin{vmatrix} u & W & V \\ W_x & v_x & U_x \\ V & U & w \end{vmatrix} + \begin{vmatrix} u & W & V \\ W & v & U \\ V_x & U_x & w_x \end{vmatrix} \quad . \qquad (147).$$

$$\Delta_{xx} = \begin{vmatrix} u_{xx} & W_{xx} & V_{xx} \\ W & v & U \\ V & U & w \end{vmatrix} + \begin{vmatrix} u & W & V \\ W_{xx} & v_{xx} & U_{xx} \\ V & U & w \end{vmatrix} + \begin{vmatrix} u & W & V \\ W & v & U \\ V_{xx} & U_{xx} & w_{xx} \end{vmatrix}$$

$$+ 2 \begin{vmatrix} u & W & V \\ W_x & v_x & U_x \\ V_x & U_x & w_x \end{vmatrix} + 2 \begin{vmatrix} u_x & W_x & V_x \\ W & v & U \\ V_x & U_x & w_x \end{vmatrix} + 2 \begin{vmatrix} u_x & W_x & V_x \\ W_x & v_x & U_x \\ V & U & w \end{vmatrix} . \qquad (148).$$

$$\Delta_{xy} = \begin{vmatrix} u_{xy} & W_{xy} & V_{xy} \\ W & v & U \\ V & U & w \end{vmatrix} + \begin{vmatrix} u & W & V \\ W_{xy} & v_{xy} & U_{xy} \\ V & U & w \end{vmatrix} + \begin{vmatrix} u & W & V \\ W & v & U \\ V_{xy} & U_{xy} & w_{xy} \end{vmatrix}$$

$$+ \left\{ \begin{vmatrix} u_y & W_y & V_y \\ W_x & v_x & U_x \\ V & U & w \end{vmatrix} + \begin{vmatrix} u_y & W_y & V_y \\ W & v & U \\ V_x & U_x & w_x \end{vmatrix} \right\}$$

$$+ \left\{ \begin{vmatrix} u_x & W_x & V_x \\ W_y & v_y & U_y \\ V & U & w \end{vmatrix} + \begin{vmatrix} u & W & V \\ W_y & v_y & U_y \\ V_x & U_x & w_x \end{vmatrix} \right\}$$

$$+ \left\{ \begin{vmatrix} u_x & W_x & V_x \\ W & v & U \\ V_y & U_y & w_y \end{vmatrix} + \begin{vmatrix} u & W & V \\ W_x & v_x & U_x \\ V_y & U_y & w_y \end{vmatrix} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (149).$$

$$\Delta_{xxx} = \begin{vmatrix} u_{xxx} & W_{xxx} & V_{xxx} \\ W & v & U \\ V & U & w \end{vmatrix} + \text{two similar terms}$$

$$+ 3 \left\{ \begin{vmatrix} u_{xx} & W_{xx} & V_{xx} \\ W_x & v_x & U_x \\ V & U & w \end{vmatrix} + \begin{vmatrix} u_{xx} & W_{xx} & V_{xx} \\ W & v & U \\ V_x & U_x & w_x \end{vmatrix} \right\} + \text{two similar terms}$$

$$+ 6 \begin{vmatrix} u_x & W_x & V_x \\ W_x & v_x & U_x \\ V_x & U_x & w_x \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (150).$$

$$\Delta_{xxy} = \begin{vmatrix} u_{xxy} & W_{xxy} & V_{xxy} \\ W & v & U \\ V & U & w \end{vmatrix} + \text{two similar terms}$$

$$+ \left\{ \begin{vmatrix} u_{xx} & W_{xx} & V_{xx} \\ W_y & v_y & U_y \\ V & U & w \end{vmatrix} + \begin{vmatrix} u_{xx} & W_{xx} & V_{xx} \\ W & v & U \\ V_y & U_y & w_y \end{vmatrix} \right\} + \text{two similar terms}$$

$$+ 2 \left\{ \begin{vmatrix} u_{xy} & W_{xy} & V_{xy} \\ W_x & v_x & U_x \\ V & U & w \end{vmatrix} + \begin{vmatrix} u_{xy} & W_{xy} & V_{xy} \\ W & v & U \\ V_x & U_x & w_x \end{vmatrix} \right\} + \text{two similar terms}$$

$$+ 2 \begin{vmatrix} u_y & W_y & V_y \\ W_x & v_x & U_x \\ V_x & U_x & w_x \end{vmatrix} + 2 \begin{vmatrix} u_x & W_x & V_x \\ W_y & v_y & U_y \\ V_x & U_x & w_x \end{vmatrix} + 2 \begin{vmatrix} u_x & W_x & V_x \\ W_x & v_x & U_x \\ V_y & U_y & w_y \end{vmatrix} . \quad . \quad (151).$$

$$\Delta_{xyz} = \begin{vmatrix} u_{xyz} & W_{xyz} & V_{xyz} \\ W & v & U \\ V & U & w \end{vmatrix} + \text{two similar terms}$$

$$+ \begin{vmatrix} u_y & W_y & V_y \\ W_x & v_x & U_x \\ V_z & U_z & w_z \end{vmatrix} + \begin{vmatrix} u_y & W_y & V_y \\ W_z & v_z & U_z \\ V_x & U_x & w_x \end{vmatrix} + \begin{vmatrix} u_z & W_z & V_z \\ W_x & v_x & U_x \\ V_y & U_y & w_y \end{vmatrix}$$

$$+ \begin{vmatrix} u_z & W_z & V_z \\ W_y & v_y & U_y \\ V_x & U_x & w_x \end{vmatrix} + \begin{vmatrix} u_x & W_x & V_x \\ W_y & v_y & U_y \\ V_z & U_z & w_z \end{vmatrix} + \begin{vmatrix} u_x & W_x & V_x \\ W_z & v_z & U_z \\ V_y & U_y & w_y \end{vmatrix}$$

$$+ \left\{ \begin{array}{l} + \begin{vmatrix} u_{xy} & W_{xy} & V_{xy} \\ W_z & v_z & U_z \\ V & U & w \end{vmatrix} + \begin{vmatrix} u_{xy} & W_{xy} & V_{xy} \\ W & v & U \\ V_z & U_z & w_z \end{vmatrix} + \begin{vmatrix} u_z & W_z & V_z \\ W_{xy} & v_{xy} & U_{xy} \\ V & U & w \end{vmatrix} \\ + \begin{vmatrix} u & W & V \\ W_{xy} & v_{xy} & U_{xy} \\ V_z & U_z & w_z \end{vmatrix} + \begin{vmatrix} u_z & W_z & V_z \\ W & v & U \\ V_{xy} & U_{xy} & w_{xy} \end{vmatrix} + \begin{vmatrix} u & W & V \\ W_z & v_z & U_z \\ V_{xy} & U_{xy} & w_{xy} \end{vmatrix} \end{array} \right\}$$

+ six determinants, which can be obtained from the last six by interchanging x and z

+ six determinants, which can be obtained from the same six determinants by interchanging y and z . . . . (152).

Art. 17.—*The relations which hold good at points on the Locus of Ultimate Intersections.*

(A.) The analytical condition (76) which holds, becomes with the notation of this section

$$uv - W^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (153).$$

Hence the values of a, b given in (145) are either infinite or indeterminate. Excluding the cases where they are infinite, it is necessary to have

$$\left.\begin{array}{l} WU - vV = 0 \\ WV - uU = 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (154).$$

Again, by substituting from (143) and (144) in (142), it follows that

$$Va + Ub + w = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (155).$$

Solving (144) and (155) for a, b, it follows that

$$\left.\begin{array}{l} a = \dfrac{vw - U^2}{WU - Vv} \\[2ex] b = \dfrac{UV - Ww}{WU - Vv} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (156).$$

Hence by (154) these values will be infinite unless

$$\left.\begin{array}{l} vw - U^2 \quad = 0 \\ UV - Ww = 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (157).$$

Hence by (153), (154), (157)

$$\left.\begin{array}{l} \quad u : W : V \\ = W : v : U \\ = V : U : w \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (158).$$

Now if $P = 0$, $Q = 0$ represent any two of the five equations (153), (154), (157), then these are satisfied at every point of the locus of ultimate intersections.

Let ξ, η, ζ and $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$ be neighbouring points on the locus of ultimate intersections.

Then

$$\frac{\partial P}{\partial \xi}\,\delta\xi + \frac{\partial P}{\partial \eta}\,\delta\eta + \frac{\partial P}{\partial \zeta}\,\delta\zeta = 0,$$

$$\frac{\partial Q}{\partial \xi}\,\delta\xi + \frac{\partial Q}{\partial \eta}\,\delta\eta + \frac{\partial Q}{\partial \zeta}\,\delta\zeta = 0$$

Now the only relation between $\delta\xi$, $\delta\eta$, $\delta\zeta$ is that which expresses that the point $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$ is on the tangent plane to the locus of intersections at ξ, η, ζ.

Hence

$$\frac{\partial P}{\partial \xi}\bigg/\frac{\partial Q}{\partial \xi} = \frac{\partial P}{\partial \eta}\bigg/\frac{\partial Q}{\partial \eta} = \frac{\partial P}{\partial \zeta}\bigg/\frac{\partial Q}{\partial \zeta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (159).$$

It is now possible to determine the values of a, b which are indeterminate as given by (145).

For, representing the value of a in (145) by the equation

$$a = P/Q,$$

it follows that the true value of a is the limit to which the expression

$$\frac{P + \frac{\partial P}{\partial \xi}\delta\xi + \frac{\partial P}{\partial \eta}\delta\eta + \frac{\partial P}{\partial \zeta}\delta\zeta}{Q + \frac{\partial Q}{\partial \xi}\delta\xi + \frac{\partial Q}{\partial \eta}\delta\eta + \frac{\partial Q}{\partial \zeta}\delta\zeta}$$

approaches, when $\delta\xi$, $\delta\eta$, $\delta\zeta$ vanish.

Now $P = 0$, $Q = 0$; hence by (159) the true value of a is equal to any one of the three ratios in (159).

Besides the values of a, b given in (145), (156), other forms may be obtained from equations (143), (155).

Putting these together

$$\left.\begin{aligned} a &= \frac{WU - vV}{uv - W^2} = \frac{vw - U^2}{WU - Vv} = \frac{UV - Ww}{WV - Uu} \\ b &= \frac{WV - uU}{uv - W^2} = \frac{UV - Ww}{WU - Vv} = \frac{uw - V^2}{WV - Uu} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (160).$$

All these values are indeterminate.

Now although the value of each of these fractions can be found by differentiating numerator and denominator with regard to any the same variable, yet they will not all lead to the true value of a, b, because the true values of a, b are found by solving the equations $ua + Wb + V = 0$, $Wa + vb + U = 0$, and finding what the values approach to as the coordinates approximate to the coordinates of a point on the locus of ultimate intersections. Now at points not on the locus of ultimate intersections, the values of a, b do not satisfy $Va + Ub + w = 0$. Hence the true values of a, b cannot in general be found by solving this last and either of the preceding equations, and then finding the values to which these approach as the coordinates approximate to the coordinates of a point on the locus of ultimate intersections.

The true values are obtainable only from the solutions (145).

(B.) If there be a conic node locus, then, besides equation (142), the following must be satisfied—

$$u_x a^2 + 2W_x ab + v_x b^2 + 2V_x a + 2U_x b + w_x = 0 \quad . \quad . \quad . \quad . \quad (161),$$

$$u_y a^2 + 2W_y ab + v_y b^2 + 2V_y a + 2U_y b + w_y = 0 \quad . \quad . \quad . \quad . \quad (162),$$

$$u_z a^2 + 2W_z ab + v_z b^2 + 2V_z a + 2U_z b + w_z = 0 \quad . \quad . \quad . \quad . \quad (163).$$

(C.) If there be a biplanar node locus, then the equations (126) are satisfied as well as the preceding.

In this case these are

$$\left.\begin{array}{l} 2(u_x a + W_x b + V_x) : 2(u_y a + W_y b + V_y) : 2(u_z a + W_z b + V_z) : 2u : 2W \\ = 2(W_x a + v_x b + U_x) : 2(W_y a + v_y b + U_y) : 2(W_z a + v_z b + U_z) : 2W : 2v \end{array}\right\} . \quad (164).$$

From these, the following may be deduced.

Introducing a quantity λ, such that

$$u_x a + W_x b + V_x = \lambda u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (165),$$

it follows by (164) that

$$W_x a + v_x b + U_x = \lambda W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (166).$$

From (165) and (166)

$$u_x a^2 + 2W_x ab + v_x b^2 + aV_x + bU_x = \lambda(au + bW).$$

Hence by (143) and (161)

$$V_x a + U_x b + w_x = \lambda V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (167).$$

Similarly, quantities μ, ν exist, such that

$$\left.\begin{array}{l} u_y a + W_y b + V_y = \mu u \\ W_y a + v_y b + U_y = \mu W \\ V_y a + U_y b + w_y = \mu V \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (168),$$

and

$$\left.\begin{array}{l} u_z a + W_z b + V_z = \nu u \\ W_z a + v_z b + U_z = \nu W \\ V_z a + U_z b + w_z = \nu V \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (169).$$

Consider now the equations (143), (144), (155), (165), (166), (167); multiply (143) by $-v_x$, (144) by W_x, (165) by $-v$, (166) by W, and add.

Therefore

$$a(2WW_x - uv_x - vu_x) - Vv_x + UW_x - vV_x + WU_x = \lambda(W^2 - uv).$$

Hence, at points on the biplanar node locus,

$$a\frac{\partial}{\partial x}(uv - W^2) = \frac{\partial}{\partial x}(UW - Vv) \quad . \quad . \quad . \quad . \quad . \quad . \quad (170).$$

Again, multiplying the same equations by W_x, $-u_x$, W, $-u$ in order and adding, it follows that

$$b\frac{\partial}{\partial x}(uv - W^2) = \frac{\partial}{\partial x}(VW - Uu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (171).$$

Again, multiplying (144), (155), (166), (167) by $-U_x$, v_x, $-U$, v in order and adding, it follows that

$$a(-WU_x + Vv_x - UW_x + vV_x) + (vw_x + wv_x - 2UU_x) = \lambda(vV - UW).$$

Hence

$$a\frac{\partial}{\partial x}(UW - vV) = \frac{\partial}{\partial x}(vw - U^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (172).$$

Again, multiplying the same equations by V_x, $-W_x$, V, $-W$ in order and adding, it follows that

$$b\frac{\partial}{\partial x}(UW - vV) = \frac{\partial}{\partial x}(UV - wW) \quad . \quad . \quad . \quad . \quad . \quad . \quad (173).$$

Again, multiplying (143), (155), (165), (167) by U_x, $-W_x$, U, $-W$ in order and adding, it follows that

$$a(uU_x - VW_x + Uu_x - WV_x) + (VU_x - wW_x + UV_x - Ww_x) = \lambda(uU - VW).$$

Therefore

$$a\frac{\partial}{\partial x}(VW - uU) = \frac{\partial}{\partial x}(UV - wW) \quad . \quad . \quad . \quad . \quad . \quad . \quad (174).$$

Again, multiplying the same equations by V_x, $-u_x$, V, $-u$ in order and adding, it follows that

$$b\frac{\partial}{\partial x}(VW - uU) = \frac{\partial}{\partial x}(uw - V^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (175).$$

Further comparing the three equations (168) or the three equations (169) with equations (165)–(167), it is evident that it is possible in any one of the equations (170)–(175) to replace x by either y or z.

It will be noticed that in the case of the biplanar node locus, the true values of the parameters may be found from any one of the ratios in (160).

(D.) If there be a uniplanar node locus, then in addition to the results obtained in (B) and (C), it follows by (48) that

$$u_{xx}a^2 + \ldots : u_{xy}a^2 + \ldots : u_{xz}a^2 + \ldots : 2(u_x a + W_x b + V_x) : 2(W_x a + v_x b + U_x)$$

$$= u_{xy}a^2 + \ldots : u_{yy}a^2 + \ldots : u_{yz}a^2 + \ldots : 2(u_y a + W_y b + V_y) : 2(W_y a + v_y b + U_y)$$

$$= u_{xz}a^2 + \ldots : u_{yz}a^2 + \ldots : u_{zz}a^2 + \ldots : 2(u_z a + W_z b + V_z) : 2(W_z a + v_z b + U_z)$$

$$= 2(u_x a + W_x b + V_x) : 2(u_y a + W_y b + V_y) : 2(u_z a + W_z b + V_z) : 2u : 2W$$

$$= 2(W_x a + v_x b + U_x) : 2(W_y a + v_y b + U_y) : 2(W_z a + v_z b + U_z) : 2W : 2v \quad (176).$$

(E.) (i.) The following equations will be useful in the case of biplanar and uniplanar node loci :—

$$\begin{vmatrix} p & R & Q \\ W_x & v_x & U_x \\ V & U & w \end{vmatrix} + \begin{vmatrix} p & R & Q \\ W & v & U \\ V_x & U_x & w_x \end{vmatrix} + \begin{vmatrix} u_x & W_x & V_x \\ R & q & P \\ V & U & w \end{vmatrix}$$

$$+ \begin{vmatrix} u & W & V \\ R & q & P \\ V_x & U_x & w_x \end{vmatrix} + \begin{vmatrix} u_x & W_x & V_x \\ W & v & U \\ Q & P & r \end{vmatrix} + \begin{vmatrix} u & W & V \\ W_x & v_x & U_x \\ Q & P & r \end{vmatrix}$$

$$= (pa^2 + 2Rab + qb^2 + 2Qa + 2Pb + r)\frac{\partial}{\partial x}(uv - W^2) \quad . \quad . \quad . \quad (177).$$

For the first and second determinants

$$= p\frac{\partial}{\partial x}(vw - U^2) + R\frac{\partial}{\partial x}(UV - Ww) + Q\frac{\partial}{\partial x}(WU - Vv).$$

The third and fourth determinants

$$= R\frac{\partial}{\partial x}(UV - Ww) + q\frac{\partial}{\partial x}(uw - V^2) + P\frac{\partial}{\partial x}(WV - Uu).$$

The fifth and sixth determinants

$$= Q\frac{\partial}{\partial x}(UW - Vv) + P\frac{\partial}{\partial x}(VW - Uu) + r\frac{\partial}{\partial x}(uv - W^2).$$

Hence in the case of biplanar and uniplanar node loci, these six determinants are by (170)–(175)

$$= (pa^2 + 2Rab + qb^2 + 2Qa + 2Pb + r)\frac{\partial}{\partial x}(uv - W^2).$$

(ii.)

$$\begin{vmatrix} u_x & W_x & V_x \\ W_x & v_x & U_x \\ V_x & U_x & w_x \end{vmatrix} = -\lambda^2 u \frac{\partial}{\partial x}(uv - W^2) \quad . \quad . \quad . \quad . \quad (178)$$

For

$$\begin{vmatrix} u_x & W_x & V_x \\ W_x & v_x & U_x \\ V_x & U_x & w_x \end{vmatrix} = \begin{vmatrix} u_x & W_x & au_x + bW_x + V_x \\ W_x & v_x & aW_x + bv_x + U_x \\ V_x & U_x & aV_x + bU_x + w_x \end{vmatrix}$$

$$= \lambda \begin{vmatrix} u_x & W_x & u \\ W_x & v_x & W \\ V_x & U_x & V \end{vmatrix}$$

$$= \lambda \begin{vmatrix} u_x & W_x & u \\ W_x & v_x & W \\ au_x + bW_x + V_x & aW_x + bv_x + U_x & au + bW + V \end{vmatrix}$$

$$= \lambda^2 \begin{vmatrix} u_x & W_x & u \\ W_x & v_x & W \\ u & W & 0 \end{vmatrix}$$

$$= \lambda^2(-u_xW^2 + 2uWW_x - u^2v_x)$$

$$= \lambda^2(-u_xuv + 2uWW_x - u^2v_x)$$

$$= -\lambda^2 u \frac{\partial}{\partial x}(uv - W^2).$$

(iii.)

$$\begin{vmatrix} u_y & W_y & V_y \\ W_x & v_x & U_x \\ V_z & U_z & w_z \end{vmatrix} + \begin{vmatrix} u_y & W_y & V_y \\ W_z & v_z & U_z \\ V_x & U_x & w_x \end{vmatrix} + \begin{vmatrix} u_z & W_z & V_z \\ W_x & v_x & U_x \\ V_y & U_y & w_y \end{vmatrix}$$

$$+ \begin{vmatrix} u_z & W_z & V_z \\ W_y & v_y & U_y \\ V_x & U_x & w_x \end{vmatrix} + \begin{vmatrix} u_x & W_x & V_x \\ W_y & v_y & U_y \\ V_z & U_z & w_z \end{vmatrix} + \begin{vmatrix} u_x & W_x & V_x \\ W_z & v_z & U_z \\ V_y & U_y & w_y \end{vmatrix}$$

$$= -2\mu\nu u \frac{\partial}{\partial x}(uv - W^2) - 2\nu\lambda u \frac{\partial}{\partial y}(uv - W^2) - 2\lambda\mu u \frac{\partial}{\partial z}(uv - W^2) \quad . \ (179).$$

For multiplying the first column in each of these determinants by a, the second by b, and adding to the last they become

$$\begin{vmatrix} u_y & W_y & \mu u \\ W_x & v_x & \lambda W \\ V_z & U_z & \nu V \end{vmatrix} + \begin{vmatrix} u_y & W_y & \mu u \\ W_z & v_z & \nu W \\ V_x & U_x & \lambda V \end{vmatrix} + \begin{vmatrix} u_z & W_z & \nu u \\ W_x & v_x & \lambda W \\ V_y & U_y & \mu V \end{vmatrix}$$

$$+ \begin{vmatrix} u_z & W_z & \nu u \\ W_y & v_y & \mu W \\ V_x & U_x & \lambda V \end{vmatrix} + \begin{vmatrix} u_x & W_x & \lambda u \\ W_y & v_y & \mu W \\ V_z & U_z & \nu V \end{vmatrix} + \begin{vmatrix} u_x & W_x & \lambda u \\ W_z & v_z & \nu W \\ V_y & U_y & \mu V \end{vmatrix}$$

The coefficient of λ is

$$\begin{vmatrix} u_y & W_y & 0 \\ W_z & v_z & W \\ V_z & U_z & V \end{vmatrix} + \begin{vmatrix} u_z & W_z & u \\ W_z & v_z & W \\ V_y & U_y & 0 \end{vmatrix} + \begin{vmatrix} u_z & W_z & u \\ W_y & v_y & 0 \\ V_z & U_z & V \end{vmatrix}$$

The coefficient of μ is

$$\begin{vmatrix} u_x & W_x & u \\ W_x & v_x & W \\ V_z & U_z & 0 \end{vmatrix} + \begin{vmatrix} u_x & W_x & u \\ W_z & v_z & 0 \\ V_x & U_x & V \end{vmatrix} + \begin{vmatrix} u_z & W_z & 0 \\ W_x & v_x & W \\ V_x & U_x & V \end{vmatrix}$$

The coefficient of ν is

$$\begin{vmatrix} u_y & W_y & u \\ W_x & v_x & 0 \\ V_y & U_y & V \end{vmatrix} + \begin{vmatrix} u_y & W_y & u \\ W_y & v_y & W \\ V_x & U_x & 0 \end{vmatrix} + \begin{vmatrix} u_x & W_x & 0 \\ W_y & v_y & W \\ V_y & U_y & V \end{vmatrix}$$

The coefficient of μ can be obtained from that of λ by changing z into x and y into z.

The coefficient of ν can be obtained from that of λ by changing y into x and z into y.

Hence it is sufficient to calculate the coefficient of λ.

The coefficient of λ, viz. :—

$$\begin{vmatrix} u_y & W_y & 0 \\ W_z & v_z & W \\ V_z & U_z & V \end{vmatrix} + \begin{vmatrix} u_z & W_z & u \\ W_y & v_y & 0 \\ V_z & U_z & V \end{vmatrix} + \begin{vmatrix} u_z & W_z & u \\ W_z & v_z & W \\ V_y & U_y & 0 \end{vmatrix}$$

$$= \begin{vmatrix} u_y & W_y & 0 \\ W_z & v_z & W \\ V_z & U_z & V \end{vmatrix} + \begin{vmatrix} u_z & W_z & u \\ W_y & v_y & 0 \\ V_z & U_z & V \end{vmatrix} + \begin{vmatrix} u_z & W_z & u \\ W_z & v_z & W \\ \mu u - a u_y - b W_y & \mu W - a W_y - b v_y & 0 \end{vmatrix}$$

$$= \mu \begin{vmatrix} u_z & W_z & u \\ W_z & v_z & W \\ u & W & 0 \end{vmatrix}$$
$$+ u_y \{ V v_z - W U_z - a (W W_z - u v_z) \}$$
$$+ W_y \{ W V_z - 2 V W_z + u U_z - b (W W_z - u v_z) - a (u W_z - W u_z) \}$$
$$+ v_y \{ V u_z - u V_z - b (u W_z - W u_z) \}$$

$$= - \mu u \frac{\partial}{\partial z} (uv - W^2)$$
$$+ u_y \{ v_z (au + V) - W (a W_z + U_z) \}$$
$$+ W_y \{ W (a u_z + V_z) - W_z (au + bW + 2V) + u (b v_z + U_z) \}$$
$$+ v_y \{ u_z (bW + V) - u (b W_z + V_z) \}$$

$$= - \mu u \frac{\partial}{\partial z} (uv - W^2)$$
$$+ u_y (- \nu W^2) + W_y (2 \nu u W) + v_y (- \nu u^2)$$

$$= - \mu u \frac{\partial}{\partial z} (uv - W^2) - \nu u \frac{\partial}{\partial y} (uv - W^2).$$

Hence the coefficient of μ obtained from this by changing z into x, and y into z; and, therefore, ν into λ, and μ into ν, is

$$- \nu u \frac{\partial}{\partial x} (uv - W^2) - \lambda u \frac{\partial}{\partial z} (uv - W^2).$$

And the coefficient of ν, obtained by changing, in the coefficient of λ, y into x, and z into y, and therefore, μ into λ, and ν into μ, is

$$- \lambda u \frac{\partial}{\partial y} (uv - W^2) - \mu u \frac{\partial}{\partial x} (uv - W^2).$$

From these the equation (179) follows.

Art. 18.—*To prove that under the conditions stated at the head of this Section, every Surface of the System touches the Locus of Ultimate Intersections along a Curve.*

Consider the surface (142), the values of a, b being now supposed to be fixed.

Consider any point ξ, η, ζ on the curve in which the surface (143) meets the locus of ultimate intersections; then, by (158), these coordinates also satisfy the surfaces (144), (155).

Multiplying (143) by a, (144) by b, (155) by 1, and adding, it follows that these coordinates also satisfy (142).

Hence any point on the curve of intersection of (143) with the locus of ultimate intersections lies on (142) and (144) also.

Hence the surfaces represented by the three fundamental equations meet the locus of ultimate intersections in the same curve.

It is necessary to prove that the surface of the system (142) will touch the locus of ultimate intersections along this curve.

Now,

$$\frac{D}{Dx}(ua^2 + 2Wab + vb^2 + 2Va + 2Ub + w)$$

$$= \frac{D}{Dx}\left[\frac{1}{u}\{(ua + Wb + V)^2 + b^2(uv - W^2) + 2b(Uu - VW) + (uw - V^2)\}\right]$$

$$= -\frac{u_x}{u^2}\{(ua + Wb + V)^2 + b^2(uv - W^2) + 2b(Uu - VW) + (uw - V^2)\}$$

$$+ \frac{1}{u}\left\{2(ua + Wb + V)\frac{D}{Dx}(ua + Wb + V)\right.$$

$$\left. + b^2\frac{D}{Dx}(uv - W^2) + 2b\frac{D}{Dx}(Uu - VW) + \frac{D}{Dx}(uw - V^2)\right\}.$$

Hence, at a point on the locus of ultimate intersections, this is equal to

$$\frac{1}{u}\left\{b^2\frac{\partial}{\partial x}(uv - W^2) + 2b\frac{\partial}{\partial x}(Uu - VW) + \frac{\partial}{\partial x}(uw - V^2)\right\}.$$

Hence the tangent plane to the surface at the point x, y, z is

$$(X - x)\left\{b^2\frac{\partial}{\partial x}(uv - W^2) + 2b\frac{\partial}{\partial x}(Uu - VW) + \frac{\partial}{\partial x}(uw - V^2)\right\}$$

$$+ (Y - y)\left\{b^2\frac{\partial}{\partial y}(uv - W^2) + 2b\frac{\partial}{\partial y}(Uu - VW) + \frac{\partial}{\partial y}(uw - V^2)\right\}$$

$$+ (Z - z)\left\{b^2\frac{\partial}{\partial z}(uv - W^2) + 2b\frac{\partial}{\partial z}(Uu - VW) + \frac{\partial}{\partial z}(uw - V^2)\right\} = 0.$$

Now this, by (159), reduces to

$$(\mathrm{X}-x)\frac{\partial}{\partial x}(uv-\mathrm{W}^2)+(\mathrm{Y}-y)\frac{\partial}{\partial y}(uv-\mathrm{W}^2)+(\mathrm{Z}-z)\frac{\partial}{\partial z}(uv-\mathrm{W}^2)=0,$$

which is the equation of the tangent plane to the locus of ultimate intersections, since $uv - \mathrm{W}^2 = 0$ at every point of the locus of ultimate intersections.

Hence each surface of the system touches the locus of ultimate intersections along a curve.

Art. 19.—*To prove that under the conditions stated at the head of this Section, there are in general at every point of the Locus of Ultimate Intersections two Conic Nodes; and if* $\mathrm{C} = 0$ *be the equation of the Locus of these Conic Nodes,* Δ *contains* C^2 *as a factor.*

(A.) To prove that there are in general two conic nodes it is necessary to show that there are in general two distinct sets of values of a, b, which satisfy (142), (161), (162), (163).

These will be satisfied if (143), (161), (162), (163) be satisfied.

Eliminating b from (143) and (161) the result is

$$a^2(\mathrm{W}^2u_x-2u\mathrm{W}\mathrm{W}_x+v_xu^2)+2a(u\mathrm{V}v_x-\mathrm{V}\mathrm{W}\mathrm{W}_x+\mathrm{W}^2\mathrm{V}_x-\mathrm{W}u\mathrm{U}_x)$$
$$+(\mathrm{V}^2v_x-2\mathrm{W}\mathrm{V}\mathrm{U}_x+\mathrm{W}^2w_x)=0.$$

Hence by (153), (154), (157), after division by u, it follows that

$$a^2\frac{\partial}{\partial x}(uv-\mathrm{W}^2)+2a\frac{\partial}{\partial x}(\mathrm{V}v-\mathrm{U}\mathrm{W})+\frac{\partial}{\partial x}(vw-\mathrm{U}^2)=0 \quad . \quad . \quad (180).^*$$

And in like manner by eliminating a between the above equations

$$b^2\frac{\partial}{\partial x}(uv-\mathrm{W}^2)+2b\frac{\partial}{\partial x}(\mathrm{U}u-\mathrm{V}\mathrm{W})+\frac{\partial}{\partial x}(uw-\mathrm{V}^2)=0 \quad . \quad . \quad (181).^\dagger$$

Further, by means of (159), it is possible in these equations to change x into y or into z.

These equations will be called the parametric quadratics.

Hence choosing a and b to satisfy (143) and (161), they will also satisfy (143) and (162), and (143) and (163).

Hence it is possible in general to find two distinct systems of values of a and b which satisfy (142), (161), (162) and (163), at points on the locus of ultimate intersections.

* The mean of the values of a satisfying (180) is the value of a given by (170).

† The mean of the values of b satisfying (181) is the value of b given by (171).

Hence there are in general two conic nodes at every point of the locus of ultimate intersections.

(B.) It follows from (146) and (147) by means of (158) that Δ, $\partial\Delta/\partial x$ both vanish at points on the locus of ultimate intersections.

By symmetry $\partial\Delta/\partial y$, $\partial\Delta/\partial z$ also vanish.

Hence Δ contains C^2 as a factor.

Example 11.—*Locus of two Conic Nodes.*

Let the surfaces be

$$(\alpha\zeta + \delta^2x^2)(x-a)^2 + 2(\beta\zeta + \delta\epsilon xy)(x-a)(y-b) + (\gamma\zeta + \epsilon^2y^2)(y-b)^2 \\ + 2g\zeta(x-a) + 2h\zeta(y-b) + k\zeta^n = 0,$$

where $\zeta = z - cx - dy$; and α, β, γ, δ, ϵ, c, d, g, h, k are fixed constants; a, b the arbitrary parameters; $n = 1$ or 2.

(A.) *The Discriminant.*

This can be formed by solving the equations

$$(\alpha\zeta + \delta^2x^2)(x-a) + (\beta\zeta + \delta\epsilon xy)(y-b) + g\zeta = 0,$$
$$(\beta\zeta + \delta\epsilon xy)(x-a) + (\gamma\zeta + \epsilon^2y^2)(y-b) + h\zeta = 0,$$

for a, b; and substituting in

$$g\zeta(x-a) + h\zeta(y-b) + k\zeta^n.$$

The values of a, b are (after removing the factor ζ which makes them indeterminate) given by

$$x - a = \frac{(h\beta - g\gamma)\zeta + \epsilon y(h\delta x - g\epsilon y)}{(\alpha\gamma - \beta^2)\zeta + (\alpha\epsilon^2y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2x^2)},$$

$$y - b = \frac{(g\beta - h\alpha)\zeta - \delta x(h\delta x - g\epsilon y)}{(\alpha\gamma - \beta^2)\zeta + (\alpha\epsilon^2y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2x^2)}.$$

Substituting these values, and multiplying by the rationalising factor

$$uv - w^2 = \zeta[(\alpha\gamma - \beta^2)\zeta + (\alpha\epsilon^2y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2x^2)]$$

the result is

$$k(\alpha\gamma - \beta^2)\zeta^{n+2} + k(\alpha\epsilon^2y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2x^2)\zeta^{n+1} \\ - (\alpha h^2 - 2\beta gh + \gamma g^2)\zeta^3 - (h\delta x - g\epsilon y)^2\zeta^2.$$

This might also have been obtained from the form (146).

(B.) *The Locus of two Conic Nodes is* $\zeta = 0$, *and if* $n = 2$ *there is also a Curve Locus of Conic Nodes.*

The singular points are determined by finding solutions of $f = 0$, $\mathrm{D}f/\mathrm{D}x = 0$, $\mathrm{D}f/\mathrm{D}y = 0$, $\mathrm{D}f/\mathrm{D}z = 0$.

But since $\mathrm{D}f/\mathrm{D}\zeta = \mathrm{D}f/\mathrm{D}z$, the equations

$$f = 0,\ \mathrm{D}f/\mathrm{D}x = 0,\ \mathrm{D}f/\mathrm{D}y = 0,\ \mathrm{D}f/\mathrm{D}\zeta = 0$$

may be used instead, where x, y, ζ are now the independent variables, so that the meaning of the symbol of differentiation D is changed.

The equations to be satisfied are

$$(\alpha\zeta + \delta^2x^2)(x - a)^2 + 2(\beta\zeta + \delta\epsilon xy)(x - a)(y - b) + (\gamma\zeta + \epsilon^2y^2)(y - b)^2 + 2g\zeta(x - a) + 2h\zeta(y - b) + k\zeta^n = 0,$$

$$[\delta x(x - a) + \epsilon y(y - b)]\,\delta(2x - a) + \zeta[\alpha(x - a) + \beta(y - b) + g] = 0,$$

$$[\delta x(x - a) + \epsilon y(y - b)]\,\epsilon(2y - b) + \zeta[\beta(x - a) + \gamma(y - b) + h] = 0,$$

$$\alpha(x - a)^2 + 2\beta(x - a)(y - b) + \gamma(y - b)^2 + 2g(x - a) + 2h(y - b) + nk\zeta^{n-1} = 0.$$

From these it follows that

$$[\delta x(x - a) + \epsilon y(y - b)]^2 + k(1 - n)\zeta^n = 0.$$

(i.) One method of solving the above equations is to take

$$\zeta = 0,$$

$$\delta x(x - a) + \epsilon y(y - b) = 0,$$

$$\alpha(x - a)^2 + 2\beta(x - a)(y - b) + \gamma(y - b)^2 + 2g(x - a) + 2h(y - b) + nk\zeta^{n-1} = 0.$$

Hence whether $n = 1$ or 2, there are two values of b, and two corresponding values of a. Hence there are two conic nodes. Hence $\zeta = 0$ is a locus of two conic nodes.

(ii.) Another method of solving the equations is to take

$$\frac{\delta(2x - a)}{\epsilon(2y - b)} = \frac{\alpha(x - a) + \beta(y - b) + g}{\beta(x - a) + \gamma(y - b) + h},$$

$$[\delta x(x - a) + \epsilon y(y - b)]\,\delta(2x - a) + \zeta[\alpha(x - a) + \beta(y - b) + g] = 0,$$

$$\alpha(x - a)^2 + 2\beta(x - a)(y - b) + \gamma(y - b)^2 + 2g(x - a) + 2h(y - b) + nk\zeta^{n-1} = 0,$$

$$[\delta x(x - a) + \epsilon y(y - b)]^2 + k(1 - n)\zeta^n = 0.$$

If $n = 1$, then $\delta x(x-a) + \epsilon y(y-b) = 0$.

Hence $\zeta = 0$, and this is the same solution as in (i.).

If, however, $n = 2$, there are four equations to be satisfied by x, y, ζ. Eliminating x, y, ζ it is necessary that a certain relation should be satisfied by a, b, in order that the equations may be consistent.

From the above equations (when $n = 2$)

$$\delta x(x-a) + \epsilon y(y-b) = \sqrt{\kappa}\,\zeta$$
$$\sqrt{\kappa}\,\delta(2x-a) + \alpha(x-a) + \beta(y-b) + g = 0,$$

rejecting a solution $\zeta = 0$.

$$\sqrt{\kappa}\,\epsilon(2y-b) + \beta(x-a) + \gamma(y-b) + h = 0,$$

$$\alpha(x-a)^2 + 2\beta(x-a)(y-b) + \gamma(y-b)^2 + 2g(x-a) + 2h(y-b) + 2\sqrt{\kappa}[\delta x(x-a) + \epsilon y(y-b)] = 0.$$

Hence

$$g(x-a) + h(y-b) + \sqrt{\kappa}[\delta a(x-a) + \epsilon b(y-b)] = 0.$$

Hence

$$\begin{aligned}
&(x-a)(\alpha + 2\delta\sqrt{\kappa}) + (y-b)\beta + (g + a\delta\sqrt{\kappa}) = 0,\\
&(x-a)\beta + (y-b)(\gamma + 2\epsilon\sqrt{\kappa}) + (h + b\epsilon\sqrt{\kappa}) = 0,\\
&(x-a)(g + a\delta\sqrt{\kappa}) + (y-b)(h + b\epsilon\sqrt{\kappa}) = 0.
\end{aligned}$$

Hence

$$\begin{vmatrix} \alpha + 2\delta\sqrt{\kappa} & \beta & g + a\delta\sqrt{\kappa} \\ \beta & \gamma + 2\epsilon\sqrt{\kappa} & h + b\epsilon\sqrt{\kappa} \\ g + a\delta\sqrt{\kappa} & h + b\epsilon\sqrt{\kappa} & 0 \end{vmatrix} = 0,$$

i.e.,

$$(g + a\,\delta\sqrt{\kappa})^2(\gamma + 2\epsilon\sqrt{\kappa}) - 2\beta(g + a\,\delta\sqrt{\kappa})(h + b\epsilon\sqrt{\kappa}) + (h + b\epsilon\sqrt{\kappa})^2(\alpha + 2\,\delta\sqrt{\kappa}) = 0.$$

Hence only when this relation holds between a, b, will there be any conic node on the surface which is not also on the locus $\zeta = 0$.

As in the general theory explained in Art. 3, this leads to a curve locus of conic nodes. It need not therefore be further considered.

Hence the only locus of conic nodes that need be considered in the discussion of the discriminant is $\zeta = 0$.

Now, whether $n = 1$ or 2, the lowest power of ζ in the discriminant is ζ^2; hence this factor is accounted for.

(C.) *The Locus*

$$k\zeta^n(\alpha\gamma - \beta^2) + k\zeta^{n-1}(\alpha\epsilon^2y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2x^2)$$
$$- \zeta(\alpha h^2 - 2\beta gh + \gamma g^2) - (h\delta x - g\epsilon y)^2 = 0$$

is an Ordinary Envelope.

The condition $uv - w^2 = 0$,

i.e.,

$$\zeta[(\alpha\gamma - \beta^2)\zeta + (\alpha\epsilon^2y^2 - 2\beta\,\delta\epsilon xy + \gamma\,\delta^2x^2)] = 0,$$

is not fulfilled at every point of this locus.

To prove that it is an envelope it will be sufficient to show that if x, y, ζ, be chosen so that

$$\begin{aligned}(\alpha\zeta + \delta^2x^2)(x - a) + (\beta\zeta + \delta\epsilon xy)(y - b) + g\zeta &= 0,\\ (\beta\zeta + \delta\epsilon xy)(x - a) + (\gamma\zeta + \epsilon^2y^2)(y - b) + h\zeta &= 0,\\ g\zeta(x - a) + h\zeta(y - b) + k\zeta^n &= 0,\end{aligned}$$

then the surface

$$k\zeta^n(\alpha\gamma - \beta^2) + k\zeta^{n-1}(\alpha\epsilon^2y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2x^2)$$
$$- \zeta(\alpha h^2 - 2\beta gh + \gamma g^2) - (h\delta x - g\epsilon y)^2 = 0$$

touches the surface

$$(\alpha\zeta + \delta^2x^2)(x - a)^2 + 2(\beta\zeta + \delta\epsilon xy)(x - a)(y - b) + (\gamma\zeta + \epsilon^2y^2)(y - b)^2$$
$$+ 2g\zeta(x - a) + 2h\zeta(y - b) + k\zeta^n = 0.$$

Calling the last two equations $\phi = 0$, $f = 0$ respectively, the conditions for contact may be expressed thus.

The same values of x, y, ζ, must satisfy

$$\phi = 0, \quad f = 0$$

$$\frac{D\phi}{Dx} \Big/ \frac{Df}{Dx} = \frac{D\phi}{Dy} \Big/ \frac{Df}{Dy} = \frac{D\phi}{D\zeta} \Big/ \frac{Df}{D\zeta},$$

where x, y, ζ are the independent variables.

The values chosen for x, y, ζ obviously make $f = 0$.

Also eliminating $x - a$, $y - b$, the result is $\zeta^2\phi = 0$.

Hence the values of x, y, ζ can be chosen so as to make $\phi = 0$.

Next

$$\begin{aligned}\frac{Df}{Dx} &= 2\delta(x - a)[\delta x(x - a) + \epsilon y(y - b)]\\ &\quad + 2[(\alpha\zeta + \delta^2x^2)(x - a) + (\beta\zeta + \delta\epsilon xy)(y - b) + g\zeta]\\ &= 2\,\delta(x - a)[\delta x(x - a) + \epsilon y(y - b)]\end{aligned}$$

for the above values of x, y, ζ.

Similarly

$$\frac{Df}{Dy} = 2\epsilon(y-b)\left[\delta x(x-a)+\epsilon y(y-b)\right],$$

$$\frac{D\phi}{Dx} = 2\kappa\zeta^{n-1}(\gamma\delta^2 x - \beta\delta\epsilon y) - 2h\delta(h\delta x - g\epsilon y),$$

$$\frac{D\phi}{Dy} = 2k\zeta^{n-1}(\alpha\epsilon^2 y - \beta\delta\epsilon x) + 2g\epsilon(h\delta x - g\epsilon y).$$

Hence

$$\frac{D\phi}{Dx}\Big/\frac{Df}{Dx} = \frac{D\phi}{Dy}\Big/\frac{Df}{Dy},$$

if

$$(x-a)\left[k\zeta^{n-1}(\alpha\epsilon y - \beta\delta x) + g(h\delta x - g\epsilon y)\right]$$
$$= (y-b)\left[k\zeta^{n-1}(\gamma\delta x - \beta\epsilon y) - h(h\delta x - g\epsilon y)\right].$$

Making use of the values of $x-a$, $y-b$, which satisfy the equations which have been taken to determine them, and which are solved above in (A), it is necessary to show that

$$\left[(h\beta - g\gamma)\zeta + \epsilon y(h\delta x - g\epsilon y)\right]\left[k\zeta^{n-1}(\alpha\epsilon y - \beta\delta x) + g(h\delta x - g\epsilon y)\right]$$
$$= \left[(g\beta - h\alpha)\zeta - \delta x(h\delta x - g\epsilon y)\right]\left[k\zeta^{n-1}(\gamma\delta x - \beta\epsilon y) - h(h\delta x - g\epsilon y)\right],$$

i.e., to show that

$$k\zeta^{n}\left[(h\beta - g\gamma)(\alpha\epsilon y - \beta\delta x) - (g\beta - h\alpha)(\gamma\delta x - \beta\epsilon y)\right]$$
$$+ k\zeta^{n-1}(h\delta x - g\epsilon y)\left[\alpha\epsilon^2 y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2 x^2\right]$$
$$- \zeta(h\delta x - g\epsilon y)(\alpha h^2 - 2\beta gh + \gamma g^2) - (h\delta x - g\epsilon y)^3 = 0,$$

i.e.,

$$(h\delta x - g\epsilon y)\phi = 0.$$

Hence this is satisfied.

Therefore

$$\frac{D\phi}{Dx}\Big/\frac{Df}{Dx} = \frac{D\phi}{Dy}\Big/\frac{Df}{Dy}.$$

It remains to prove that each of the equal quantities

$$\frac{(x-a)\left[\delta x(x-a)+\epsilon y(y-b)\right]}{k\zeta^{n-1}(\gamma\delta x - \beta\epsilon y) - h(h\delta x - g\epsilon y)},\quad \frac{(y-b)\left[\delta x(x-a)+\epsilon y(y-b)\right]}{k\zeta^{n-1}(\alpha\epsilon y - \beta\delta x) + g(h\delta x - g\epsilon y)}$$

is equal to

$$\frac{\alpha(x-a)^2 + 2\beta(x-a)(y-b) + \gamma(y-b)^2 + 2g(x-a) + 2h(y-b) + nk\zeta^{n-1}}{nk\zeta^{n-1}(\alpha\gamma - \beta^2) + (n-1)k\zeta^{n-2}(\alpha\epsilon^2 y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2 x^2) - (\alpha h^2 - 2\beta gh + \gamma g^2)}.$$

Multiply numerator and denominator of the first ratio by δx, of the second by ϵy, of the third by ζ, and form a new ratio by addition of the numerators and deno minators.

Then each of these ratios must be equal to

$$\frac{\alpha\zeta(x-a)^2+2\beta\zeta(x-a)(y-b)+\gamma\zeta(y-b)^2+2g\zeta(x-a)+2h\zeta(y-b)+nk\zeta^n+[\delta x(x-a)+\epsilon y(y-b)]^2}{nk\zeta^n(\alpha\gamma-\beta^2)+nk\zeta^{n-1}(\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2)-\zeta(\alpha h^2-2\beta gh+\gamma g^2)-(h\delta x-g\epsilon y)^2}.$$

Hence, by means of the equations $f=0$, $\phi=0$, each of the ratios must be equal to

$$(n-1)k\zeta^n/[(n-1)k\zeta^n(\alpha\gamma-\beta^2)+(n-1)k\zeta^{n-1}(\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2)]$$

i.e.,

$$\zeta/[\zeta(\alpha\gamma-\beta^2)+(\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2)].$$

Hence it will be sufficient to prove

$$\frac{(x-a)[\delta x(x-a)+\epsilon y(y-b)]}{k\zeta^{n-1}(\gamma\delta x-\beta\epsilon y)-h(h\delta x-g\epsilon y)}=\frac{\zeta}{\zeta(\alpha\gamma-\beta^2)+(\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2)}.$$

Now using the values of $x-a$, $y-b$ given above in (A),

$$[\delta x(x-a)+\epsilon y(y-b)][\zeta(\alpha\gamma-\beta^2)+(\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2)]$$
$$=\zeta[\delta x(h\beta-g\gamma)+\epsilon y(g\beta-h\alpha)].$$

Hence, using the value of $(x-a)$, it is necessary to prove that

$$[(h\beta-g\gamma)\zeta+\epsilon y(h\delta x-g\epsilon y)][\delta x(h\beta-g\gamma)+\epsilon y(g\beta-h\alpha)]$$
$$=[(\alpha\gamma-\beta^2)\zeta+(\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2)][k\zeta^{n-1}(\gamma\delta x-\beta\epsilon y)-h(h\delta x-g\epsilon y)].$$

Hence it is necessary to show that

$$(\gamma\delta x-\beta\epsilon y)[k\zeta^n(\alpha\gamma-\beta^2)+k\zeta^{n-1}(\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2)]$$
$$=\zeta[h(\alpha\gamma-\beta^2)(h\delta x-g\epsilon y)-g(h\beta-g\gamma)(\gamma\delta x-\beta\epsilon y)+h(h\beta-g\gamma)(\beta\delta x-\alpha\epsilon y)]$$
$$+(h\delta x-g\epsilon y)[h(\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2)-g\epsilon y(\gamma\delta x-\beta\epsilon y)+h\epsilon y(\beta\delta x-\alpha\epsilon y)],$$

i.e.,

$$(\gamma\delta x-\beta\epsilon y)\phi=0.$$

Hence this is satisfied.

Hence the conditions for contact are satisfied.

Since $uv-w^2=0$ is not satisfied at all points of the locus $\phi=0$, the factor of the discriminant corresponding to it occurs only once.

(D.) It will be verified that the mean of the values of the parameter b, which correspond to the two surfaces having conic nodes, at a point on the locus $\zeta = 0$, is the same as the value of the parameter b, which is used to form the discriminant.

The values of the parameters corresponding to the conic node are given by

$$\zeta = 0, \quad \delta x\,(x - a) + \epsilon y\,(y - b) = 0,$$
$$\alpha\,(x - a)^2 + 2\beta\,(x - a)\,(y - b) + \gamma\,(y - b)^2 + 2g\,(x - a) + 2h\,(y - b) + nk\zeta^{n-1} = 0.$$

Hence

$$(y - b)^2\,(\alpha\epsilon^2 y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2 x^2) + 2\,(y - b)\,\delta x\,(h\delta x - g\epsilon y) + nk\delta^2 x^2 \zeta^{n-1} = 0.$$

Hence the mean of the values of $y - b$ is

$$\delta x\,(g\epsilon y - h\delta x)/(\alpha\epsilon^2 y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2 x^2).$$

Now putting $\zeta = 0$ in the value of $y - b$, given above in (A), the same result is obtained.

(E.) This example is a case in which the assumption *equivalent* to that of Art. 7, viz., that

$$\frac{\mathrm{D}f}{\mathrm{D}a}\,\frac{\partial a}{\partial z} + \frac{\mathrm{D}f}{\mathrm{D}b}\,\frac{\partial b}{\partial z} = 0$$

at points on the locus of ultimate intersections cannot be made.

The equations $\mathrm{D}f/\mathrm{D}a = 0$, $\mathrm{D}f/\mathrm{D}b = 0$ are given in (A).

Hence $\partial a/\partial z$, $\partial b/\partial z$ are given by

$$(\alpha\zeta + \delta^2 x^2)\frac{\partial a}{\partial z} + (\beta\zeta + \delta\epsilon xy)\frac{\partial b}{\partial z} = \alpha\,(x - a) + \beta\,(y - b) + g,$$
$$(\beta\zeta + \delta\epsilon xy)\frac{\partial a}{\partial z} + (\gamma\zeta + \epsilon^2 y^2)\frac{\partial b}{\partial z} = \beta\,(x - a) + \gamma\,(y - b) + h.$$

Denoting for brevity

$\alpha\,(x - a) + \beta\,(y - b) + g$ by G,

$\beta\,(x - a) + \gamma\,(y - b) + h$ by H,

$\zeta^2\,(\alpha\gamma - \beta^2) + \zeta\,(\alpha\epsilon^2 y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2 x^2)$ by K,

$\delta x\,(x - a) + \epsilon y\,(y - b)$ by L,

it follows that

$$\frac{\partial a}{\partial z} = \frac{1}{\mathrm{K}}[\mathrm{G}\,(\gamma\zeta + \epsilon^2 y^2) - \mathrm{H}\,(\beta\zeta + \delta\epsilon xy)],$$
$$\frac{\partial b}{\partial z} = \frac{1}{\mathrm{K}}[\mathrm{H}\,(\alpha\zeta + \delta^2 x^2) - \mathrm{G}\,(\beta\zeta + \delta\epsilon xy)].$$

Therefore

$$-\tfrac{1}{2}\left(\frac{\mathrm{D}f}{\mathrm{D}a}\frac{\partial a}{\partial z}+\frac{\mathrm{D}f}{\mathrm{D}b}\frac{\partial b}{\partial z}\right)$$

$$=\frac{1}{\mathrm{K}}(\mathrm{G}\zeta+\mathrm{L}\delta x)\left[\mathrm{G}(\gamma\zeta+\epsilon^2y^2)-\mathrm{H}(\beta\zeta+\delta\epsilon xy)\right]$$

$$+\frac{1}{\mathrm{K}}(\mathrm{H}\zeta+\mathrm{L}\epsilon y)\left[\mathrm{H}(\alpha\zeta+\delta^2x^2)-\mathrm{G}(\beta\zeta+\delta\epsilon xy)\right].$$

Hence, dividing numerator and denominator by ζ, and then putting $\zeta=0$,

$$-\tfrac{1}{2}\left(\frac{\mathrm{D}f}{\mathrm{D}a}\frac{\partial a}{\partial z}+\frac{\mathrm{D}f}{\mathrm{D}b}\frac{\partial b}{\partial z}\right)=\frac{(\mathrm{G}\epsilon y-\mathrm{H}\delta x)^2+\mathrm{L}\delta x(\mathrm{G}\gamma-\mathrm{H}\beta)+\mathrm{L}\epsilon y(\mathrm{H}\alpha-\mathrm{G}\beta)}{\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2}.$$

Now, in the case $n=2$, there is a conic node when $x=a$, $y=b$, $\zeta=0$, and then $\mathrm{G}=g$, $\mathrm{H}=h$, $\mathrm{L}=0$.

Hence

$$-\tfrac{1}{2}\left(\frac{\mathrm{D}f}{\mathrm{D}a}\frac{\partial a}{\partial z}+\frac{\mathrm{D}f}{\mathrm{D}b}\frac{\partial b}{\partial z}\right)=\frac{(g\epsilon b-h\delta a)^2}{\alpha\epsilon^2b^2-2\beta\delta\epsilon ab+\gamma\delta^2a^2}.$$

Hence

$$\frac{\mathrm{D}f}{\mathrm{D}a}\frac{\partial a}{\partial z}+\frac{\mathrm{D}f}{\mathrm{D}b}\frac{\partial b}{\partial z}$$

does not vanish.

Art. 20.—*To prove that under the conditions stated at the head of this Section, if the two Surfaces having Conic Nodes coincide, then they are replaced by a single Surface having a Biplanar or a Uniplanar Node.*

If the condition be expressed that the roots of either parametric quadratic be equal, then the roots of the other parametric quadratic must also in general be equal; for treating the parameters as coordinates of points in a plane, this amounts to expressing that the straight line (143) touches the conic (161).

In this case then, the two surfaces having conic nodes coincide, and if a, b be the values of the parameters corresponding to them, they may be found by finding the points of contact of the straight line (143) with the conic (161).

They are therefore given by the equations

$$\frac{u_xa+\mathrm{W}_xb+\mathrm{V}_x}{u}=\frac{\mathrm{W}_xa+v_xb+\mathrm{U}_x}{\mathrm{W}}=\frac{\mathrm{V}_xa+\mathrm{U}_xb+w_x}{\mathrm{V}}.$$

Now, since the equation (161) may be replaced by (162) or (163), it follows that in the above x may be changed into y or z. Hence

$$u_xa+\mathrm{W}_xb+\mathrm{V}_x : u_ya+\mathrm{W}_yb+\mathrm{V}_y : u_za+\mathrm{W}_zb+\mathrm{V}_z : u \;:\mathrm{W}$$
$$=\mathrm{W}_xa+v_xb+\mathrm{U}_x : \mathrm{W}_ya+v_yb+\mathrm{U}_y : \mathrm{W}_za+v_zb+\mathrm{U}_z : \mathrm{W} : v.$$

Hence the conditions (164) are satisfied.

Hence there is in general a biplanar node.

But as a particular case there may be a uniplanar node.

Art. 21.—*If the two Conic Nodes are replaced by a single Biplanar Node, and if* $B = 0$ *be the equation of the Biplanar Node Locus, and if the Edge of the Biplanar Node touch the Biplanar Node Locus,* Δ *contains* B^3 *as a factor.*

It follows as in Art. 19 (B.) that Δ, $\partial\Delta/\partial x$, $\partial\Delta/\partial y$, $\partial\Delta/\partial z$ all vanish on the biplanar node locus.

Consider now $\partial^2\Delta/\partial x^2$ as given in (148).

The first three determinants vanish by (158).

To calculate the next three, put in (177)

$$p = u_x,\ q = v_x,\ r = w_x,\ P = U_x,\ Q = V_x,\ R = W_x.$$

Hence these three determinants

$$= (a^2 u_x + 2ab W_x + b^2 v_x + 2a V_x + 2b U_x + w_x) \frac{\partial}{\partial x} (uv - W^2) = 0$$

by (161).

Next consider $\partial^2\Delta/\partial x\,\partial y$ as given in (149).

The first three determinants vanish by (158).

To obtain the next six, put in (177),

$$p = u_y,\ q = v_y,\ r = w_y,\ P = U_y,\ Q = V_y,\ R = W_y.$$

Hence their value is

$$(a^2 u_y + 2ab W_y + b^2 v_y + 2a V_y + 2b U_y + w_y) \frac{\partial}{\partial x} (uv - W^2) = 0$$

by (162).

Hence by symmetry all the second differential coefficients of Δ vanish.

Therefore Δ contains B^3 as a factor.

Example 12.—*Locus of Biplanar Nodes, such that the Edges of the Biplanar Nodes always touch the Biplanar Node Locus, the equation of the Surfaces of the System being of the Second Degree in the Parameters.*

Let the surfaces be

$$(bx - ay + cz)^2 - g^2 z^2 - 2mz (x - a) (y - b) = 0,$$

where c, g, m are fixed constants; a, b are the arbitrary parameters.

(A.) *The Discriminant.*

$$\Delta = \begin{vmatrix} y^2 & -xy - mz & (m-c)\,yz \\ -xy - mz & x^2 & (m+c)\,xz \\ (m-c)\,yz & (m+c)\,xz & (c^2-g^2)\,z^2 - 2m\,xyz \end{vmatrix}$$

$$= mz^3\,\{2g^2xy - (c^2-g^2)\,mz\}.$$

The way in which the factor z^3 arises will now be examined.

The discriminant is found by eliminating a, b between

$$(bx - ay + cz)^2 - g^2z^2 - 2m\,(x-a)\,(y-b)\,z = 0 \quad . \quad . \quad . \quad . \quad (\alpha),$$

$$-\,2y\,(bx - ay + cz) + 2m\,(y-b)\,z = 0 \quad . \quad . \quad . \quad . \quad (\beta),$$

$$2x\,(bx - ay + cz) + 2m\,(x-a)\,z = 0 \quad . \quad . \quad . \quad . \quad (\gamma).$$

By means of (β), (γ), it follows that (α) can be written

$$(bx - ay + cz)\,cz - g^2z^2 - mz\,(2xy - bx - ay) = 0 \quad . \quad . \quad . \quad . \quad (\delta).$$

The values of a, b, satisfying (β) and (γ) are

$$\frac{a}{\begin{vmatrix} -xy - mz & (m-c)\,yz \\ x^2 & (m+c)\,xz \end{vmatrix}} = \frac{b}{\begin{vmatrix} (m-c)\,yz & y^2 \\ (m+c)\,xz & -xy - mz \end{vmatrix}} = \frac{1}{\begin{vmatrix} y^2 & -xy - mz \\ -xy - mz & x^2 \end{vmatrix}}$$

Therefore

$$\frac{a}{-\,mxz\,\{2xy + (m+c)\,z\}} = \frac{b}{-\,myz\,\{2xy + (m-c)\,z\}} = \frac{1}{-\,mz\,(2xy + mz)}.$$

Now it will be shown that on the binode locus $z = 0$; therefore the values of a, b become indeterminate on the binode locus.

But they may be determined by dividing out by the factor z, which vanishes on the binode locus, and then

$$a = x\left(1 + \frac{cz}{2xy + mz}\right),$$

$$b = y\left(1 - \frac{cz}{2xy + mz}\right).$$

Hence if ξ, η, 0 be any point on the binode locus, then at this point the values of the parameters are $a = \xi$, $b = \eta$.

Hence there is a single set of values of the parameters satisfying the equations $\mathrm{D}f/\mathrm{D}a = 0$, $\mathrm{D}f/\mathrm{D}b = 0$ at points on the binode locus, which has been determined.

There is not a double set of equal values as in Art. 15 (see especially Example 10 of that article), where the degree of the equation of the system of surfaces in the parameters is higher than the second.

If the values given above for a, b be substituted in the left-hand side of (δ), and the result multiplied by the rationalising factor $uv - \mathrm{W}^2$, which in this case is $-(2xymz + m^2z^2)$, the result is

$$-(2xymz + m^2z^2)\left\{\frac{mc^2z^3}{2xy + mz} - g^2z^2\right\} = mz^3\{2g^2xy - (c^2 - g^2)mz\},$$

which is the same value for the discriminant as before.

It will be noticed that the factor z enters once through the rationalising factor, and twice from the remaining part.

(B.) *The Node Locus is* $z = 0$.

Substituting $x = a + \mathrm{X}$, $y = b + \mathrm{Y}$, $z = \mathrm{Z}$ in the equation, it becomes

$$(b\mathrm{X} - a\mathrm{Y} + c\mathrm{Z})^2 - g^2\mathrm{Z}^2 - 2m\mathrm{XYZ} = 0.$$

Hence the new origin is a binode. There are no other singular points on the surface.

The biplanes are $b\mathrm{X} - a\mathrm{Y} + c\mathrm{Z} \pm g\mathrm{Z} = 0$.

They intersect in the straight line $b\mathrm{X} - a\mathrm{Y} = 0$, $\mathrm{Z} = 0$.

Hence the binode locus is $z = 0$, and the edge of the binode, which lies in the binode locus, satisfies the condition for contact with the binode locus.

(C.) *The Locus* $(c^2 - g^2)mz - 2g^2xy = 0$ *is an Ordinary Envelope.*

To prove this it is necessary to satisfy at the same time

$$(bx - ay + cz)^2 - g^2z^2 - 2mz(x - a)(y - b) = 0 \quad . \quad . \quad . \quad . \quad (\alpha),$$

$$(c^2 - g^2)mz - 2g^2xy = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\epsilon),$$

$$\frac{2b(bx - ay + cz) - 2m(y - b)z}{-2g^2y} = \frac{-2a(bx - ay + cz) - 2m(x - a)z}{-2g^2x}$$
$$= \frac{2c(bx - ay + cz) - 2g^2z - 2m(x - a)(y - b)}{m(c^2 - g^2)} \quad . \quad . \quad . \quad (\zeta).$$

Multiplying numerator and denominator of the first ratio in (ζ) by x, of the second by y, and of the third by z; adding the numerators to form a new numerator, and the denominators to form a new denominator, and reducing by (α) and (ϵ), each of the above ratios

$$= \frac{2(xy - ab)}{c^2 - g^2},$$

Equating the third ratio of (ζ) to this, and substituting for z from (ϵ), and putting $a/x = \xi$, $b/y = \eta$, the result is

$$m(m-c)\,\xi + m(m+c)\,\eta + 2(g^2 - m^2) = 0 \quad . \quad . \quad . \quad . \quad . \quad (\eta).$$

In like manner, from the first and third ratios of (ζ),

$$m^2(c^2 - g^2)\,\eta^2 + 4m(m+c)\,g^2\eta + 4(g^4 - m^2g^2) \\ + \xi\,\{-m^2(c^2+g^2)\,\eta + 2mg^2(m-c)\} = 0 \quad . \quad . \quad . \quad (\theta).$$

Substituting for ξ from equation (η), this reduces to

$$mc\eta^2 + \eta\,(g^2 - cm) = 0.$$

Hence $\eta = 0$, $\eta = 1 - (g^2/cm)$.
Substituting in equation (η) the corresponding values of ξ are

$$\xi = \frac{2(m^2 - g^2)}{m(m-c)}, \quad \xi = 1 + (g^2/cm).$$

It remains to prove that one of the two systems of solutions will satisfy the equation obtained from the second and third ratios of (ζ).

This equation is

$$m^2(c^2 - g^2)\,\xi^2 + 4m(m-c)\,g^2\xi + 4(g^4 - m^2g^2) \\ + \eta\,[-m^2(c^2+g^2)\,\xi + 2mg^2(m+c)] = 0.$$

Substituting for η from equation (η), this reduces to

$$mc\xi^2 - \xi\,(cm + g^2) = 0.$$

Therefore

$$\xi = 0, \quad \xi = 1 + (g^2/cm).$$

Hence the solutions

$$x = acm/(cm + g^2),$$
$$y = bcm/(cm - g^2),$$
$$z = 2abc^2g^2m/\{(c^2m^2 - g^4)(c^2 - g^2)\},$$

satisfy all the equations.

Hence the surface (ϵ) is an envelope.

(D.) It will now be verified that the values of b given by (181), and the equations obtained by changing ξ into η and ζ, become equal in this case.

The equation

$$b^2 \frac{\partial}{\partial \zeta}(uv - \mathrm{W}^2) + 2b \frac{\partial}{\partial \zeta}(u\mathrm{U} - \mathrm{VW}) + \frac{\partial}{\partial \zeta}(uw - \mathrm{V}^2) = 0$$

is the only one that need be considered, because the others are identically satisfied.

In this case

$$uv - \mathrm{W}^2 = -2xymz - m^2z^2$$
$$u\mathrm{U} - \mathrm{VW} = myz\,\{2xy + (m - c)\,z\}$$
$$uw - \mathrm{V}^2 = (-m^2 + 2mc - g^2)\,y^2z^2 - 2mxy^3z.$$

Hence if $\xi, \eta, 0$ be any point on the binode locus,

$$\frac{\partial}{\partial \zeta}(uv - \mathrm{W}^2) = -2m\xi\eta,$$
$$\frac{\partial}{\partial \zeta}(u\mathrm{U} - \mathrm{VW}) = 2m\xi\eta^2,$$
$$\frac{\partial}{\partial \zeta}(uw - \mathrm{V}^2) = -2m\xi\eta^3.$$

Hence the equation for b is

$$-2m\xi\eta\,(b - \eta)^2 = 0.$$

Hence both values of b become equal to η.

Art. 22.—*If the two Conic Nodes are replaced by a single Uniplanar Node, and if* $\mathrm{U} = 0$ *be the equation of the Uniplanar Node Locus, then* Δ *contains* U^4 *as a factor.*

It follows, as in Art. 19 (B), and Art. 21, that Δ and all its differential coefficients of the second order vanish.

Next take the value of $\partial^3\Delta/\partial x^3$ from (150).

The first three determinants vanish by (158).

To calculate the next set of terms, put in (177)

$$p = u_{xx}, \quad q = v_{xx}, \quad r = w_{xx}, \quad \mathrm{P} = \mathrm{U}_{xx}, \quad \mathrm{Q} = \mathrm{V}_{xx}, \quad \mathrm{R} = \mathrm{W}_{xx}.$$

Hence they are equal to

$$3\,(a^2u_{xx} + 2ab\mathrm{W}_{xx} + b^2v_{xx} + 2a\mathrm{V}_{xx} + 2b\mathrm{U}_{xx} + w_{xx})\frac{\partial}{\partial x}(uv - \mathrm{W}^2)$$
$$= 6\lambda^2u\frac{\partial}{\partial x}(uv - \mathrm{W}^2).$$

The last determinant by (178) is equal to

$$- 6\lambda^2 u \frac{\partial}{\partial x}(uv - W^2).$$

Hence $\partial^3\Delta/\partial x^3 = 0$.

Next take the value of $\partial^3\Delta/\partial x^2\partial y$ from (151).
The first three terms vanish by (158).
The next three terms by (177) are equal to

$$(a^2 u_{xx} + 2ab W_{xx} + b^2 v_{xx} + 2a V_{xx} + 2b U_{xx} + w_{xx}) \frac{\partial}{\partial y}(uv - W^2)$$
$$= 2\lambda^2 u \frac{\partial}{\partial y}(uv - W^2).$$

The next three terms by (177) are equal to

$$2(a^2 u_{xy} + 2ab W_{xy} + b^2 v_{xy} + 2a V_{xy} + 2b U_{xy} + w_{xy}) \frac{\partial}{\partial x}(uv - W^2)$$
$$= 4\lambda\mu u \frac{\partial}{\partial x}(uv - W^2).$$

The next three terms may be obtained from (179) by changing z into x, and, therefore, ν into λ. They are therefore equal to

$$- 4\lambda\mu u \frac{\partial}{\partial x}(uv - W^2) - 2\lambda^2 u \frac{\partial}{\partial y}(uv - W^2).$$

Hence $\partial^3\Delta/\partial x^2\partial y = 0$.

Next take $\partial^3\Delta/\partial x\, \partial y\, \partial z$ from (152).
The first three determinants vanish by (158).
The next six are by (179)

$$= - 2\mu\nu u \frac{\partial}{\partial x}(uv - W^2) - 2\nu\lambda u \frac{\partial}{\partial y}(uv - W^2) - 2\lambda\mu u \frac{\partial}{\partial z}(uv - W^2).$$

The next six are by (177)

$$= (a^2 u_{xy} + 2ab W_{xy} + b^2 v_{xy} + 2a V_{xy} + 2b U_{xy} + w_{xy}) \frac{\partial}{\partial z}(uv - W^2)$$
$$= 2\lambda\mu u \frac{\partial}{\partial z}(uv - W^2).$$

Hence, the next six are

$$= 2\mu\nu u \frac{\partial}{\partial x}(uv - W),$$

and the next six are

$$= 2\nu\lambda u \frac{\partial}{\partial y}(uv - W^2).$$

Hence, $\partial^3\Delta/\partial x\,\partial y\,\partial z = 0$.

Hence, by symmetry, all the third differential coefficients of Δ vanish.
Hence, Δ contains U^4 as a factor.

Example 13.—*Locus of Uniplanar Nodes when the equation of the System of Surfaces is of the Second Degree in the Parameters.*

Let the surfaces be

$$(bx - ay + z)^2 - z^3 - 2mz(x - a)(y - b) = 0.$$

(A.) *The Discriminant.*

It is

$$\begin{vmatrix} y^2 & -xy - mz & (m-1)yz \\ -xy - mz & x^2 & (m+1)xz \\ (m-1)yz & (m+1)xz & z^2 - z^3 - 2mxyz \end{vmatrix}$$
$$= mz^4(2xy + mz - m).$$

To show the origin of the factor z^4, the formation of the discriminant will be examined.

The equations $Df/Da = 0$, $Df/Db = 0$ are, in this case

$$ay^2 - b(xy + mz) + (m-1)yz = 0,$$
$$-a(xy + mz) + bx^2 + (m+1)xz = 0.$$

Therefore

$$\frac{a}{-mxz\{2xy + (m+1)z\}} = \frac{b}{-myz\{2xy + (m-1)z\}} = \frac{1}{-mz(2xy + mz)}.$$

Now, it will be shown presently that $z = 0$ is the uniplanar node locus. Hence, a, b become indeterminate on the uniplanar node locus. But, removing the factor $-mz$, which vanishes on this locus,

$$a = x\left(1 + \frac{z}{2xy + mz}\right), \quad b = y\left(1 - \frac{z}{2xy + mz}\right).$$

Hence, at any point, $\xi, \eta, 0$ on the uniplanar node locus, $a = \xi$, $b = \eta$.

Again, substituting the above values of a, b in

$$(bx - ay + z)^2 - z^3 - 2mz(x - a)(y - b),$$

the result is

$$\frac{mz^3}{2xy + mz} - z^3.$$

If this be expanded in ascending powers of z, the lowest is the third power.

But the rationalising factor applied to form the discriminant, viz. $- mz(2xy + mz)$ contains the factor z. Hence, the factor z^4 is accounted for.

The discriminant is as before

$$mz^4 (2xy + mz - m).$$

(B.) *The Uniplanar Node Locus is* $z = 0$.

Put $x = a + X$, $y = b + Y$, $z = Z$ in the equation. It becomes

$$(bX - aY + Z)^2 - Z^3 - 2mXYZ = 0.$$

Hence the new origin is a uniplanar node.

Hence $z = 0$ is the uniplanar node locus.

(C.) *The Envelope Locus is* $2xy + mz - m = 0$.

The equation can be written

$$\begin{aligned} a^2y^2 - 2ab(xy + mz) + b^2x^2 + 2a(m - 1)yz + 2b(m + 1)xz + \rho \\ = \rho - z^2 + z^3 + 2mxyz. \end{aligned}$$

Let ρ be determined as a function of x, y, z, so that the left-hand side of the equation may break up into factors linear with regard to a, b.

Then

$$\rho = z^2 - 2mxyz - \frac{mz^3}{2xy + mz}.$$

It may then be verified that the equation can be written

$$\begin{aligned} y^2\Big[\Big\{a + \frac{1}{y^2}[- b(xy + mz) + (m - 1)yz]\Big\}^2 \\ - \frac{m}{y^4}z(2xy + mz)\Big\{b - y + \frac{yz}{2xy + mz}\Big\}^2\Big] = \frac{z^3(2xy + mz - m)}{2xy + mz}. \end{aligned}$$

Hence it may be concluded that $2xy + mz - m = 0$ will touch the surface where both the factors of the left-hand side vanish, *i.e.*, where

$$a + \frac{1}{y^2}[- b(xy + mz) + (m - 1)yz] = 0,$$

$$b - y + \frac{yz}{2xy + mz} = 0,$$

i.e., where

$$a = x\left(1 + \frac{z}{2xy + mz}\right), \quad b = y\left(1 - \frac{z}{2xy + mz}\right).$$

Hence the points of contact are determined by

$$2xy + mz - m = 0,$$
$$a = x\left(1 + \frac{z}{m}\right),$$
$$b = y\left(1 - \frac{z}{m}\right).$$

Hence

$$(z^2 - m^2)(z - 1) - 2abm = 0.$$

Hence when a, b are given, there are three values of z, and three corresponding values of x, and three corresponding values of y. Hence each surface touches the envelope at three points. But each point on the envelope is the point of contact of only one surface of the system, since when the coordinates x, y, z of the point of contact are given, the values of a, b, the parameters of the surface touching the envelope there, are determined by the simple equations

$$a = x(1 + z/m), \quad b = y(1 - z/m).$$

The result may be verified thus:—

The values of x, y, z satisfying the equations

$$a = x(1 + z/m), \quad b = y(1 - z/m), \quad (z^2 - m^2)(z - 1) - 2abm = 0 \quad . \quad . \quad (\alpha),$$

will satisfy at the same time

$$\left.\begin{aligned}(bx - ay + z)^2 - z^3 - 2mz(x - a)(y - b) = 0\\ 2xy + mz - m = 0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (\beta),$$

and

$$\frac{2(bx - ay + z)b - 2mz(y - b)}{2y} = \frac{-2(bx - ay + z)a - 2mz(x - a)}{2x}$$
$$= \frac{2(bx - ay + z) - 3z^2 - 2m(x - a)(y - b)}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\gamma).$$

If x, y, z satisfy (α), then

$$bx - ay + z = \frac{z(z^2 + 2abm - m^2)}{z^2 - m^2},$$

and substituting in the first of equations (β), after making some reductions, the result is

$$\frac{z^2}{(z^2-m^2)^2}(z^2+2abm-m^2)\left[2abm-(z^2-m^2)(z-1)\right]=0,$$

which is satisfied by (α).

Hence the first of equations (β) is satisfied by the values of x, y, z given by (α).

Again substituting for x, y in terms of z from (α) in (γ) the ratios become equal to

$$\begin{aligned}&\left[z^3(-m-1)-m^2z^2+(m^2-2abm)z\right]/m(m+z)\\ =&\left[z^3(-m+1)+m^2z^2-(m^2-2abm)z\right]/m(m-z)\\ =&\left[3z^4-2z^3+z^2(2mab-3m^2)+z(2m^2-4abm)\right]/m(m^2-z^2).\end{aligned}$$

Hence it is necessary to show that

$$\begin{aligned}&\left[z^2(-m-1)-m^2z+(m^2-2abm)\right](m-z)\\ =&\left[z^2(-m+1)+m^2z-(m^2-2abm)\right](m+z)\\ =&\,3z^3-2z^2+z(2mab-3m^2)+(2m^2-4abm) \quad . \quad . \quad . \quad . \quad (\delta).\end{aligned}$$

Equating the first and second quantities in (δ) it is necessary to prove that

$$z^3-z^2-m^2z+m^2-2abm=0,$$

which holds by (α).

Equating the second and third quantities in (δ) and removing the factor ($m+2$), the same result is obtained.

Hence the values of x, y, z given in (α) satisfy all the equations (β), (γ).

Art. 23.—*If the parameters of one of the two Surfaces having Conic Nodes become infinite, and if* $C=0$ *be the equation of the Conic Node Locus,* Δ *contains* C^2 *as a factor.*

The conditions that one value of a and one value of b satisfying the parametric quadratics (180) and (181) should be infinite are that

$$\frac{\partial}{\partial x}(uv-W^2)=0,\quad \frac{\partial}{\partial y}(uv-W^2)=0,\quad \frac{\partial}{\partial z}(uv-W^2)=0.$$

In this case the values of Δ and $\partial\Delta/\partial x$, as given by (146) and (147), both vanish. Hence Δ contains C^2 as a factor.

Example 14.—*Locus of one Conic Node.*

Let the surfaces be

$$\left[\alpha(x-a)+\beta(y-b)\right]^2+2gz(x-a)+2hz(y-b)+kz^2=0.$$

(A.) *The Discriminant.*

It reduces to

$$\begin{vmatrix} \alpha^2 & \alpha\beta & gz \\ \alpha\beta & \beta^2 & hz \\ gz & hz & kz^2 \end{vmatrix} = -(h\alpha - g\beta)^2 z^2.$$

(B.) *The Conic Node Locus is* $z = 0$.

In this case equations (143), (161), (162), (163) are equivalent to the three equations

$$\alpha^2(x - a) + \alpha\beta(y - b) + gz = 0,$$
$$\alpha\beta(x - a) + \beta^2(y - b) + hz = 0,$$
$$g(x - a) + h(y - b) + kz = 0,$$

the only solutions of which (unless $g\beta - h\alpha = 0$) are

$$x = a, \quad y = b, \quad z = 0.$$

Hence there is now only one system of values of the parameters satisfying (143), (161), (162), (163).

The same value of the parameter b would be obtained from the equation (181) which becomes in this case, after changing x into ζ,

$$2b\frac{\partial}{\partial\zeta}(u\mathrm{U} - \mathrm{VW}) + \frac{\partial}{\partial\zeta}(uw - \mathrm{V}^2) = 0.$$

Now

$$u\mathrm{U} - \mathrm{VW} = (g\beta - h\alpha)\,\alpha z,$$
$$uw - \mathrm{V}^2 = 2(h\alpha - g\beta)\,\alpha yz + (k\alpha^2 - g^2)\,z^2;$$

therefore,

$$\frac{\partial}{\partial z}(u\mathrm{U} - \mathrm{VW}) = (g\beta - h\alpha)\,\alpha,$$

$$\frac{\partial}{\partial z}(uw - \mathrm{V}^2) = 2(h\alpha - g\beta)\,\alpha y + 2(k\alpha^2 - g^2)\,z.$$

On the conic node locus $z = 0$.

Therefore the equation for b is

$$2b(g\beta - h\alpha)\,\alpha + 2(h\alpha - g\beta)\,\alpha y = 0.$$

Therefore

$$b = y.$$

There is only one conic node, since $uv - \mathrm{W}^2 = 0$, and, therefore, equation (181) reduces to a simple equation for b.

Art. 24.—*If the parameters of both of the Surfaces having Conic Nodes become infinite, and* $E = 0$ *be the equation of the Envelope Locus, then* Δ *contains* E^3 *as a factor.*

In this case it is necessary that both roots of the parametric quadratics (180) and (181) should become infinite.

Hence the first differential coefficients of

$$uv - W^2, \quad Vv - UW, \quad Uu - VW,$$

with regard to any of the variables, must vanish on the envelope locus.

[It may be noted that if

$$\frac{\partial}{\partial x}(uv - W^2) = 0, \text{ and } \frac{\partial}{\partial x}(Vv - UW) = 0, \text{ then } \frac{\partial}{\partial x}(Uu - VW) = 0.$$

For

$$\left.\begin{array}{l} \frac{\partial}{\partial x}(uv - W^2) = u_x v + u v_x - 2WW_x = 0 \\ \frac{\partial}{\partial x}(Vv - UW) = V_x v + V v_x - U_x W - U W_x = 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad (182).$$

Multiplying these equations by V, u respectively, and subtracting

$$u_x V v + U_x W u - uv V_x - W_x (2WV - Uu) = 0\,;$$

therefore, using (158), after dividing by W,

$$u_x U + U_x u - W V_x - W_x V = 0\,;$$

therefore

$$\frac{\partial}{\partial x}(Uu - VW) = 0.]$$

Now Δ, $\partial\Delta/\partial x$ both vanish by (158).

Next consider $\partial^2\Delta/\partial x^2$ as given by (148).

The first three determinants vanish by (158).

The fourth and fifth determinants

$$= 2\left\{V_x \frac{\partial}{\partial x}(WU - Vv) + U_x \frac{\partial}{\partial x}(VW - Uu) + w_x \frac{\partial}{\partial x}(uv - W^2)\right\}$$

$= 0$ by the above conditions.

The sixth determinant is

$$2\begin{vmatrix} u_x & W_x & V_x \\ W_x & v_x & U_x \\ V & U & w \end{vmatrix}$$

Substituting in it for u_x, v_x the values which can be obtained from

$$\frac{\partial}{\partial x}(\mathrm{V}v - \mathrm{UW}) = 0, \text{ and } \frac{\partial}{\partial x}(\mathrm{U}u - \mathrm{VW}) = 0,$$

it becomes

$$\frac{2}{\mathrm{UV}}\begin{vmatrix} \mathrm{WV}_x + \mathrm{VW}_x - u\mathrm{U}_x & \mathrm{UW}_x & \mathrm{UV}_x \\ \mathrm{VW}_x & \mathrm{U}_x\mathrm{W} + \mathrm{UW}_x - \mathrm{V}_x v & \mathrm{VU}_x \\ \frac{\mathrm{V}}{u}u & \frac{\mathrm{V}}{u}\mathrm{W} & \frac{\mathrm{V}}{u}\mathrm{V} \end{vmatrix}$$

$$= \frac{2}{u\mathrm{U}}\begin{vmatrix} \mathrm{WV}_x + \mathrm{VW}_x - u\mathrm{U}_x & \mathrm{UW}_x & \mathrm{UV}_x \\ \mathrm{VW}_x & \mathrm{UW}_x + \mathrm{U}_x\mathrm{W} - \mathrm{V}_x v & \mathrm{VU}_x \\ u & \mathrm{W} & \mathrm{V} \end{vmatrix}$$

$$= \frac{2}{u\mathrm{U}}\begin{vmatrix} \mathrm{WV}_x + \mathrm{VW}_x & \mathrm{UW}_x + \mathrm{WU}_x & \mathrm{UV}_x + \mathrm{VU}_x \\ \mathrm{VW}_x & \mathrm{UW}_x + \mathrm{WU}_x - \mathrm{V}_x v & \mathrm{VU}_x \\ u & \mathrm{W} & \mathrm{V} \end{vmatrix}$$

$$= \frac{2}{u\mathrm{U}}\begin{vmatrix} \mathrm{WV}_x & v\mathrm{V}_x & \mathrm{UV}_x \\ \mathrm{VW}_x & \mathrm{UW}_x + \mathrm{WU}_x - \mathrm{V}_x v & \mathrm{VU}_x \\ u & \mathrm{W} & \mathrm{V} \end{vmatrix}$$

$$= \frac{2\mathrm{V}_x}{\mathrm{U}u}\begin{vmatrix} \mathrm{W} & v & \mathrm{U} \\ \mathrm{VW}_x & \mathrm{UW}_x + \mathrm{WU}_x - \mathrm{V}_x v & \mathrm{VU}_x \\ u & \mathrm{W} & \mathrm{V} \end{vmatrix}$$

$= 0$ by (158).

Next take $\partial^2\Delta/\partial x\,\partial y$ from (149).

The first three determinants vanish by (158).

The fifth and seventh determinants

$$= \mathrm{V}_x\frac{\partial}{\partial y}(\mathrm{UW} - \mathrm{V}v) + \mathrm{U}_x\frac{\partial}{\partial y}(\mathrm{VW} - \mathrm{U}u) + w_x\frac{\partial}{\partial y}(uv - \mathrm{W}^2) = 0.$$

The eighth and ninth determinants

$$= \mathrm{V}_y\frac{\partial}{\partial x}(\mathrm{UW} - \mathrm{V}v) + \mathrm{U}_y\frac{\partial}{\partial x}(\mathrm{VW} - \mathrm{U}u) + w_y\frac{\partial}{\partial x}(uv - \mathrm{W}^2) = 0.$$

The fourth and sixth determinants are

$$\begin{vmatrix} u_y & W_y & V_y \\ W_x & v_x & U_x \\ V & U & w \end{vmatrix} + \begin{vmatrix} u_x & W_x & V_x \\ W_y & v_y & U_y \\ V & U & w \end{vmatrix}$$

$$= \frac{1}{UV} \begin{vmatrix} WV_y + VW_y - uU_y & UW_y & UV_y \\ VW_x & U_xW + UW_x - V_xv & VU_x \\ \frac{V}{u}u & \frac{V}{u}W & \frac{V}{u}V \end{vmatrix}$$

$$+ \frac{1}{UV} \begin{vmatrix} WV_x + VW_x - uU_x & UW_x & UV_x \\ VW_y & U_yW + UW_y - V_yv & VU_y \\ \frac{V}{u}u & \frac{V}{u}W & \frac{V}{u}V \end{vmatrix}$$

$$= \frac{1}{Uu} \begin{vmatrix} WV_y + VW_y & UW_y + WU_y & UV_y + VU_y \\ VW_x & U_xW + UW_x - V_xv & VU_x \\ u & W & V \end{vmatrix}$$

$$+ \frac{1}{Uu} \begin{vmatrix} WV_x + VW_x & UW_x + WU_x & UV_x + VU_x \\ VW_y & U_yW + UW_y - V_yv & VU_y \\ u & W & V \end{vmatrix}$$

Hence the coefficient of U_y is

$$\frac{1}{Uu} \begin{vmatrix} 0 & W & V \\ VW_x & U_xW+UW_x-V_xv & VU_x \\ u & W & V \end{vmatrix} + \frac{1}{Uu} \begin{vmatrix} WV_x+VW_x & UW_x+WU_x & UV_x+VU_x \\ 0 & W & V \\ u & W & V \end{vmatrix}$$

$$= \frac{1}{U} \begin{vmatrix} W & V \\ UW_x - V_xv & 0 \end{vmatrix} + \frac{1}{U} \begin{vmatrix} UW_x & UV_x \\ W & V \end{vmatrix}$$

$$= \frac{1}{U}(VvV_x - VUW_x + UVW_x - UWV_x) = 0.$$

The coefficient of V_y is

$$\frac{1}{Uu}\begin{vmatrix} W & 0 & U \\ VW_x & U_xW + UW_x - V_xv & VU_x \\ u & W & V \end{vmatrix} + \frac{1}{Uu}\begin{vmatrix} WV_x + VW_x & UW_x + WU_x & UV_x + VU_x \\ 0 & -v & 0 \\ u & W & V \end{vmatrix}$$

$$= \frac{1}{Uu}\begin{vmatrix} W & 0 & U \\ VW_x & U_xW + UW_x - V_xv & VU_x \\ 0 & W & 0 \end{vmatrix} - \frac{v}{Uu}(VWV_x + V^2W_x - uUV_x - uVU_x)$$

$$= -\frac{W}{Uu}(WVU_x - UVW_x) - \frac{v}{Uu}(V^2W_x - uVU_x)$$

$$= \frac{1}{Uu}\{U_x(uv - W^2)V + W_x(UW - vV)V\} = 0.$$

The coefficient of W_y is

$$\frac{1}{Uu}\begin{vmatrix} V & U & 0 \\ VW_x & U_xW + UW_x - V_xv & VU_x \\ u & W & V \end{vmatrix}$$

$$+ \frac{1}{Uu}\begin{vmatrix} WV_x + VW_x & UW_x + WU_x & UV_x + VU_x \\ V & U & 0 \\ u & W & V \end{vmatrix}$$

$$= \frac{V^2}{Uu}(U_xW - V_xv) + \frac{V}{Uu}(UWV_x - VWU_x)$$

$$= \frac{V}{Uu}\{V_x(UW - Vv)\} = 0.$$

Hence $\partial^2\Delta/\partial x\,\partial y = 0$.

Hence all the differential coefficients of the second order vanish.

Hence Δ contains E^3 as a factor.

Example 15.—*Envelope Locus, the parameters of both the Surfaces having Conic Nodes being infinite.*

Let the surfaces be

$$z^2a^2 + z + (ax + b + y)^2 = 0.$$

(A.) *The Discriminant.*

This is

$$\begin{vmatrix} z^2 + x^2 & x & xy \\ x & 1 & y \\ xy & y & y^2 + z \end{vmatrix}$$

$$= z^3.$$

(B.) *The Envelope Locus is* $z = 0$.

The tangent plane at ξ, η, ζ is

$$(\mathrm{X} - \xi)\, 2a\,(a\xi + b + \eta) + (\mathrm{Y} - \eta)\, 2\,(a\xi + b + \eta) + (\mathrm{Z} - \zeta)\,(2a^2\zeta + 1) = 0.$$

Hence at the point ξ, η, ζ, where

$$a\xi + b + \eta = 0,\ \zeta = 0,$$

the tangent plane is $\mathrm{Z} = 0$.

Hence the factor z^3 is accounted for.

(C.) *The Parameters of both Surfaces having Conic Nodes are infinite.*

In this case

$$u = z^2 + x^2,\ v = 1,\ w = y^2 + z,\ \mathrm{U} = y,\ \mathrm{V} = xy,\ \mathrm{W} = x.$$

Hence the equations

$$a^2 \frac{\partial}{\partial z}(uv - \mathrm{W}^2) + 2a \frac{\partial}{\partial z}(\mathrm{V}v - \mathrm{UW}) + \frac{\partial}{\partial z}(vw - \mathrm{U}^2) = 0,$$

and

$$b^2 \frac{\partial}{\partial z}(uv - \mathrm{W}^2) + 2b \frac{\partial}{\partial z}(\mathrm{U}u - \mathrm{VW}) + \frac{\partial}{\partial z}(uw - \mathrm{V}^2) = 0,$$

become, when $z = 0$,

$$(0)\, a^2 + 0\,(a) + 1 = 0,$$
$$(0)\, b^2 + 0\,(b) + x^2 = 0.$$

Hence both roots are infinite.

If the differential coefficients in the parametric quadratics had been taken with regard to x or y, the equations would have been wholly indeterminate.

Art. 25.—*If the parameters of both of the Surfaces having Conic Nodes become indeterminate, then at every point of the Locus of Ultimate Intersections there are an infinite number of Biplanar Nodes; each Surface of the system has a Binodal Line lying on the Locus of Ultimate Intersections, and if the locus of these Binodal Lines be* $\mathrm{B} = 0$, *then* Δ *contains* B^4 *as a factor.*

In order that the parametric quadratics may become wholly indeterminate, the first differential coefficients, with regard to each of the three variables, of $uv - \mathrm{W}^2$, $\mathrm{U}u - \mathrm{VW}$, $\mathrm{V}v - \mathrm{UW}$, $vw - \mathrm{U}^2$, $uw - \mathrm{V}^2$ must vanish. These involve the vanishing of the first differential coefficients of $\mathrm{UV} - \mathrm{W}w$.

It will be shown, first of all, that the ratios (164) are in this case equivalent only to the equation (143). [The same holds good in the previous article, but the condition (161) is not satisfied there.]

For consider the ratios

$$u_x a + W_x b + V_x : W_x a + v_x b + U_x = u : W.$$

Therefore

$$a(Wu_x - uW_x) + b(WW_x - uv_x) + (WV_x - uU_x) = 0.$$

This will be the same equation as (143) if

$$\frac{Wu_x - uW_x}{u} = \frac{WW_x - uv_x}{W} = \frac{WV_x - uU_x}{V}.$$

Hence if

$$u_x W^2 + u^2 v_x - 2uWW_x = 0,$$

and

$$VWW_x - Vuv_x - W^2 V_x + WuU_x = 0,$$

i.e., if

$$uvu_x + u^2 v_x - 2uWW_x = 0,$$

$$UuW_x - Vuv_x - uvV_x + WuU_x = 0,$$

i.e., if

$$\frac{\partial}{\partial x}(uv - W^2) = 0,$$

$$\frac{\partial}{\partial x}(UW - Vv) = 0,$$

which are satisfied.

Similarly the other ratios in (164) hold. Hence if any point be taken on the curve in which the surface (143) intersects the locus of ultimate intersections, that point is a binode on the surface (142). Hence the surface (142) has a binodal line situated on the locus of ultimate intersections. Hence each surface of the system has a binodal line situated on the locus of ultimate intersections.

It remains to show that if $B = 0$ be the locus of these binodal lines then Δ contains B^4 as a factor.

The proof in the last article will hold as far as the second differential coefficients of Δ are concerned.

Consider, therefore, the value of $\partial^3\Delta/\partial x^3$ given in (150).

The first three terms vanish by (158).

The next three are equal to three times

$$u_{xx}\frac{\partial}{\partial x}(vw - U^2) + 2W_{xx}\frac{\partial}{\partial x}(UV - Ww) + v_{xx}\frac{\partial}{\partial x}(uw - V^2)$$
$$+ 2V_{xx}\frac{\partial}{\partial x}(WU - Vv) + 2U_{xx}\frac{\partial}{\partial x}(WV - Uu) + w_{xx}\frac{\partial}{\partial x}(uv - W^2)$$
$$= 0.$$

The last determinant is by (178)

$$= -6\lambda^2 u \frac{\partial}{\partial x}(uv - W^2) = 0.$$

Therefore $\partial^3\Delta/\partial x^3 = 0$.

Next take $\partial^3\Delta/\partial x^2\,\partial y$ as given in (151).
The first three terms vanish by (158).
The next three

$$= u_{xx}\frac{\partial}{\partial y}(vw - U^2) + 2W_{xx}\frac{\partial}{\partial y}(UV - Ww) + v_{xx}\frac{\partial}{\partial y}(uw - V^2)$$
$$+ 2V_{xx}\frac{\partial}{\partial y}(WU - Vv) + 2U_{xx}\frac{\partial}{\partial y}(WV - Uu) + w_{xx}\frac{\partial}{\partial y}(uv - W^2)$$
$$= 0.$$

The next three

$$= 2u_{xy}\frac{\partial}{\partial x}(vw - U^2) + 4W_{xy}\frac{\partial}{\partial x}(UV - Ww) + 2v_{xy}\frac{\partial}{\partial x}(uw - V^2)$$
$$+ 4V_{xy}\frac{\partial}{\partial x}(WU - Vv) + 4U_{xy}\frac{\partial}{\partial x}(WV - Uu) + 2w_{xy}\frac{\partial}{\partial x}(uv - W^2)$$
$$= 0.$$

The next three may be calculated by means of (179) by putting $z = x$, and therefore $\nu = \lambda$.
Hence they are equal to

$$-4\lambda\mu u\frac{\partial}{\partial x}(uv - W^2) - 2\lambda^2 u\frac{\partial}{\partial y}(uv - W^2) = 0.$$

Hence $\partial^3\Delta/\partial x^2\,\partial y = 0$.

Next take $\partial^3\Delta/\partial x\,\partial y\,\partial z$ as given in (152).
The first three terms vanish by (158).
The next six terms by (179)

$$= -2\mu\nu u\frac{\partial}{\partial x}(uv - W^2) - 2\nu\lambda u\frac{\partial}{\partial y}(uv - W^2) - 2\lambda\mu u\frac{\partial}{\partial z}(uv - W^2)$$
$$= 0.$$

The next six terms

$$= u_{xy}\frac{\partial}{\partial z}(vw - U^2) + 2W_{xy}\frac{\partial}{\partial z}(UV - Ww) + v_{xy}\frac{\partial}{\partial z}(uw - V^2)$$
$$+ 2V_{xy}\frac{\partial}{\partial z}(WU - Vv) + 2U_{xy}\frac{\partial}{\partial z}(WV - Uu) + w_{xy}\frac{\partial}{\partial z}(uv - W^2)$$
$$= 0.$$

The next six terms being obtainable from these last six by interchanging x and z vanish.

The remaining six vanish in like manner.

Hence all the third differential coefficients of Δ vanish.

Hence Δ contains B^4 as a factor.

Example 16.—*Locus of Binodal Lines.*

Let the surfaces be

$$z^2 \{a^2 + \phi(x, y, z)\} - (ax + b + y)^2 = 0.$$

(A.) *The Discriminant.*

This is

$$\begin{vmatrix} z^2 - x^2 & -x & -xy \\ -x & -1 & -y \\ -xy & -y & -y^2 + z^2\phi(x, y, z) \end{vmatrix} = -z^4\phi(x, y, z).$$

(B.) *The Locus of Binodal Lines is* $z = 0$.

For let ξ, η, ζ be any point on both the loci $z = 0$, $ax + b + y = 0$.

Then put $x = \xi + X$, $y = \eta + Y$, $z = \zeta + Z$, so that $\zeta = 0$, $a\xi + b + \eta = 0$.

Therefore

$$Z^2\left(a^2 + \phi(\xi, \eta, \zeta) + X\frac{\partial\phi}{\partial\xi} + Y\frac{\partial\phi}{\partial\eta} + Z\frac{\partial\phi}{\partial\zeta} + \ldots\right) - (a\xi + b + \eta + aX + Y)^2 = 0.$$

Hence the lowest terms in X, Y, Z are

$$Z^2 \{a^2 + \phi(\xi, \eta, \zeta)\} - (aX + Y)^2 = 0.$$

These break up into two factors.

Hence the point ξ, η, ζ is a binode on the surface.

Hence the straight line $z = 0$, $ax + b + y = 0$ is a binodal line on the surface.

And $z = 0$ is the locus of binodal lines.

Hence the factor z^4 of the discriminant is accounted for.

(C.) *The Locus* $\phi(x, y, z) = 0$ *is connected with a Curve Locus, not a Surface Locus, of Ultimate Intersections.*

For the fundamental equations are in this case

$$z^2 \{a^2 + \phi(x, y, z)\} - (ax + b + y)^2 = 0,$$
$$2z^2a - 2x(ax + b + y) = 0,$$
$$2(ax + b + y) = 0.$$

Hence, if $\phi(x, y, z) = 0$, then, in order that the above equations may be satisfied,

$$ax + b + y = 0,$$
$$z = 0.$$

The locus of these points is the curve

$$z = 0, \quad \phi(x, y, z) = 0.$$

This belongs to one of the exceptional cases enumerated in the Section VI. of this paper.

Example 17.—*This example shows the difference between the cases when the equation is of the Second Degree in the parameters and those in which it is of a Higher Degree, so far as regards Binode and Unode Loci.*

Let the surfaces be

$$\alpha(x - a)^3 + 3\beta(x - a)^2 z + cz^3 + 3d(y - b)^2 + ez^2 = 0.$$

(A.) *The Discriminant.*

It is the same as that of the equation

$$\alpha X^3 + 3\beta z X^2 Z + 3dY^2 Z + (cz^3 + ez^2) Z^3 = 0.$$

Therefore

$$S = -\beta^2 d^2 z^2,$$
$$T = 4d^3 z^2 \{\alpha^2 (e + cz) + 2\beta^3 z\}.$$

Therefore

$$\Delta = 16d^6 z^4 \{\alpha^4 (e + cz)^2 + 4\alpha^2 \beta^3 z (e + cz)\}.$$

(B.) *The Locus of Biplanar Nodes is $z = 0$.*

For putting $x = a + X$, $y = b + Y$, $z = Z$, the equation becomes

$$\alpha X^3 + 3\beta X^2 Z + cZ^3 + 3dY^2 + eZ^2 = 0.$$

The edge of the biplanes is given by $Y = 0$, $Z = 0$.

Hence the edge of the biplanes lies in the biplanar node locus $z = 0$, and, therefore, satisfies the condition for contact with the biplanar node locus.

Hence the factor z^4 is accounted for (Art. 15).

(C.) *If $e = 0$, the Locus of Uniplanar Nodes is $z = 0$.*

In this case,

$$\Delta = 16d^6 (\alpha^4 c^2 + 4\alpha^2 \beta^3 c) z^6.$$

Hence the factor z^6 is accounted for (Art. 12).

(D.) *If* $\alpha = 0$, Δ *appears to vanish, but then the equation of the Surfaces,*

$$3\beta(x-a)^2z + cz^3 + 3d(y-b)^2 + ez^2 = 0,$$

is of the Second Degree in the parameters, and if the Discriminant be formed it does not really vanish.

For the discriminant required is not that of the cubic

$$3\beta z X^2 Z + 3dY^2Z + (cz^3 + ez^2)Z^3,$$

but of the quadric

$$3\beta z X^2 + 3dY^2 + (cz^3 + ez^2)Z^2.$$

It is therefore

$$9\beta dz^3(cz + e).$$

(E.) *The Locus of Biplanar Nodes is now* $z = 0$, *the edge of the Biplanar Node being in the Biplanar Node Locus.*

The edge satisfies the condition for contact with the biplanar node locus. Hence the factor z^3 is accounted for (Art. 21).

(F.) *If* $e = 0$, *the Locus of Uniplanar Nodes is* $z = 0$.

In this case the discriminant is $9\beta cdz^4$. Hence the factor z^4 is accounted for (Art. 22).

Section V. (Arts. 26–29).—The Intersections of Consecutive Surfaces.

It has been shown that when the analytical condition (76) is satisfied which expresses that the fundamental equations are satisfied by two coinciding systems of values, the number of factors in the discriminant corresponding to conic node, biplanar node, and uniplanar node loci, is less when the degree of the equation in the parameters is the second than when it is of a higher degree.

It has also been shown that, when (76) holds and the degree in the parameters is the second, each surface of the system, its consecutive surfaces, and the locus of ultimate intersections, intersect in a common curve.

It is desirable, therefore, to examine the nature of the intersections of consecutive surfaces in all other cases.

Art. 26.—*To prove that the Surfaces represented by the three fundamental equations intersect in one point on the Envelope Locus, unless the Envelope Locus have stationary contact with each Surface of the System, and then there are two points of intersection.*

(A.) First consider the case of an ordinary envelope.

Let ξ, η, ζ be a point of intersection of the surfaces

$$f(x, y, z, a, b) = 0,$$
$$\frac{Df}{Da} = 0,$$
$$\frac{Df}{Db} = 0.$$

Let $\xi + X$, $\eta + Y$, $\zeta + Z$ be a neighbouring point on the same three surfaces, so that the values of a, b are the same.

Therefore

$$f + [\xi] X + [\eta] Y + [\zeta] Z + \tfrac{1}{2} \{[\xi, \xi] X^2 + \ldots\} + \tfrac{1}{6} \{[\xi, \xi, \xi] X^3 + \ldots\} + \ldots = 0 \quad (183),$$

$$[\alpha] + [\xi, \alpha] X + [\eta, \alpha] Y + [\zeta, \alpha] Z + \tfrac{1}{2} \{[\xi, \xi, \alpha] X^2 + \ldots\} + \ldots = 0 \quad (184),$$

$$[\beta] + [\xi, \beta] X + [\eta, \beta] Y + [\zeta, \beta] Z + \tfrac{1}{2} \{[\xi, \xi, \beta] X^2 + \ldots\} + \ldots = 0 \quad (185).$$

Hence because $f = 0$, $[\alpha] = 0$, $[\beta] = 0$, the terms of lowest order in X, Y, Z in (183), (184), (185) are of the first degree in each case. Hence there is one solution $X = 0$, $Y = 0$, $Z = 0$. Hence there is one intersection at this point.

(B.) Next consider the case where the contact is stationary.

The equation of the tangent plane to the envelope locus is

$$(X - \xi)[\xi] + (Y - \eta)[\eta] + (Z - \zeta)[\zeta] = 0 \quad . \quad . \quad . \quad . \quad (186).$$

But also from (28) and (29)* by means of (76) the equation of the tangent plane can also be shown to be

$$[\alpha, \beta] \{(X - \xi)[\alpha, \xi] + (Y - \eta)[\alpha, \eta] + (Z - \zeta)[\alpha, \zeta]\} - [\alpha, \alpha] \{(X - \xi)[\beta, \xi] + (Y - \eta)[\beta, \eta] + (Z - \zeta)[\beta, \zeta]\} = 0 \quad (187).$$

Hence

$$\frac{1}{[\xi]} \{[\alpha, \beta][\alpha, \xi] - [\alpha, \alpha][\beta, \xi]\} = \frac{1}{[\eta]} \{[\alpha, \beta][\alpha, \eta] - [\alpha, \alpha][\beta, \eta]\} = \frac{1}{[\zeta]} \{[\alpha, \beta][\alpha, \zeta] - [\alpha, \alpha][\beta, \zeta]\}.$$

Hence the lowest terms in (183), (184), (185) are not independent of each other; and if the three fractions last written be each $= \mu$, it is possible by multiplying (183)

* Equations (28) and (29) are satisfied at any point of an envelope locus. Equations (16), (17), (18) are not.

by μ, (184) by $-[\alpha, \beta]$, (185) by $[\alpha, \alpha]$, and adding, to form a new equation in which the lowest terms in X, Y, Z are of the second degree.

Hence the equations (183)–(185) are equivalent to three others in which the lowest terms in X, Y, Z are of degrees 2, 1, 1 respectively. Hence there are two sets of zero values of X, Y, Z. Hence there are two intersections.

Art. 27.—*To prove that the Surfaces represented by the three fundamental equations intersect in two points on the Conic Node Locus, unless it be also an Envelope Locus, and then there are three points of intersection.*

(A.) In the case of the Conic Node Locus $[\xi] = 0$, $[\eta] = 0$, $[\zeta] = 0$.

Hence the lowest terms in X, Y, Z in (183) are of the second degree, in (184) and (185) of the first degree.

Hence there are two intersections.

(B.) In the case where the conic node locus is also an envelope, it will be shown that the values of X, Y, Z, which make

$$[\alpha, \xi]\,X + [\alpha, \eta]\,Y + [\alpha, \zeta]\,Z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (188),$$

$$[\beta, \xi]\,X + [\beta, \eta]\,Y + [\beta, \zeta]\,Z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (189),$$

also make

$$[\xi, \xi]\,X^2 + [\eta, \eta]\,Y^2 + [\zeta, \zeta]\,Z^2 + 2\,[\eta, \zeta]\,YZ + 2\,[\zeta, \xi]\,ZX + 2\,[\xi, \eta]\,XY = 0 \quad (190),$$

so that the lowest terms in the equations, by which (183)–(185) may be replaced, are of degree 3, 1, 1 respectively, and hence there are three intersections.

Now the cone (190) touches the tangent plane to the conic node locus, viz. :—

$$\begin{aligned}[\alpha, \beta]\,\{[\alpha, \xi]\,X + [\alpha, \eta]\,Y + [\alpha, \zeta]\,Z\}& \\ - [\alpha, \alpha]\,\{[\beta, \xi]\,X + [\beta, \eta]\,Y + [\beta, \zeta]\,Z\} &= 0 \quad . \quad . \quad . \quad . \quad . \quad (191),\end{aligned}$$

this being the form for the tangent plane to the conic node locus which can be deduced from (28), (29), and (76).

Hence, to find the line of contact, whose equations are

$$X/X' = Y/Y' = Z/Z',$$

the origin of co-ordinates being taken at the singular point,

$$\begin{aligned}&\{[\xi, \xi]\,X' + [\xi, \eta]\,Y' + [\xi, \zeta]\,Z'\} \,/\, \{[\alpha, \beta]\,[\alpha, \xi] - [\alpha, \alpha]\,[\beta, \xi]\} \\ = &\{[\xi, \eta]\,X' + [\eta, \eta]\,Y' + [\eta, \zeta]\,Z'\} \,/\, \{[\alpha, \beta]\,[\alpha, \eta] - [\alpha, \alpha]\,[\beta, \eta]\} \\ = &\{[\xi, \zeta]\,X' + [\eta, \zeta]\,Y' + [\zeta, \zeta]\,Z'\} \,/\, \{[\alpha, \beta]\,[\alpha, \zeta] - [\alpha, \alpha]\,[\beta, \zeta]\} \quad . \quad (192);\end{aligned}$$

It will now be verified that

$$X'/\{[\alpha,\eta][\beta,\zeta]-[\alpha,\zeta][\beta,\eta]\} = Y'/\{[\alpha,\zeta][\beta,\xi]-[\alpha,\xi][\beta,\zeta]\}$$
$$= Z'/\{[\alpha,\xi][\beta,\eta]-[\alpha,\eta][\beta,\xi]\}.$$

For substituting these values in the first and second ratios of (192), they become

$$\frac{D[[\xi],[\eta],[\zeta]]}{D[\xi,\alpha,\beta]} \Big/ \frac{D[[\alpha],[\beta]]}{D[\xi,\alpha]} = \frac{D[[\xi],[\eta],[\zeta]]}{D[\eta,\alpha,\beta]} \Big/ \frac{D[[\alpha],[\beta]]}{D[\eta,\alpha]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (193).$$

Now, since the equations (16), (17), (18), (28), (29), are equivalent to three equations only, it follows that

$$\frac{D[[\xi],[\eta],[\zeta],[\alpha]]}{D[\xi,\eta,\alpha,\beta]} = 0,$$

which may be written

$$[\alpha,\beta]\frac{D[[\xi],[\eta],[\zeta]]}{D[\xi,\eta,\alpha]} - [\alpha,\alpha]\frac{D[[\xi],[\eta],[\zeta]]}{D[\xi,\eta,\beta]} + [\alpha,\eta]\frac{D[[\xi],[\eta],[\zeta]]}{D[\xi,\alpha,\beta]} - [\alpha,\xi]\frac{D[[\xi],[\eta],[\zeta]]}{D[\eta,\alpha,\beta]} = 0 \quad . \quad . \quad . \quad (194).$$

Also from (16), (17), (18), (28), (29) may be deduced

$$\frac{D[[\xi],[\eta],[\zeta],[\beta]]}{D[\xi,\eta,\alpha,\beta]} = 0,$$

which may be written

$$[\beta,\beta]\frac{D[[\xi],[\eta],[\zeta]]}{D[\xi,\eta,\alpha]} - [\alpha,\beta]\frac{D[[\xi],[\eta],[\zeta]]}{D[\xi,\eta,\beta]} + [\beta,\eta]\frac{D[[\xi],[\eta],[\zeta]]}{D[\xi,\alpha,\beta]} - [\beta,\xi]\frac{D[[\xi],[\eta],[\zeta]]}{D[\eta,\alpha,\beta]} = 0 \quad . \quad . \quad . \quad (195).$$

Multiplying (194) by $[\alpha,\beta]$, (195) by $[\alpha,\alpha]$, and subtracting,

$$\frac{D\{[\xi],[\eta],[\zeta]\}}{D\ [\xi,\alpha,\beta]}\{[\alpha,\eta][\alpha,\beta]-[\beta,\eta][\alpha,\alpha]\} = \frac{D\{[\xi],[\eta],[\zeta]\}}{D\ [\eta,\alpha,\beta]}\{[\alpha,\xi][\alpha,\beta]-[\beta,\xi][\alpha,\alpha]\},$$

which proves (193).

This proves that the first ratio of (192) is equal to the second. By symmetry the first ratio is also equal to the third.

Hence the line of contact is the intersection of the planes

$$\left.\begin{array}{l}[\alpha, \xi]X + [\alpha, \eta]Y + [\alpha, \zeta]Z = 0\\ [\beta, \xi]X + [\beta, \eta]Y + [\beta, \zeta]Z = 0\end{array}\right\} \quad \ldots \ldots \quad (196).$$

Hence the values of X, Y, Z which satisfy (188) and (189) also satisfy (190), which was to be proved.

It may be noticed that the equations (196) are those of the tangent planes to the surfaces $Df/D\alpha = 0$, $Df/D\beta = 0$ at ξ, η, ζ.

Art. 28.—*To prove that the Surfaces represented by the three fundamental equations intersect in three points on the Biplanar Node Locus, unless the Edge of the Biplanar Node always touch the Biplanar Node Locus, and then there are four points of intersection.*

(A.) In this case

$$[\xi, \xi]X^2 + [\eta, \eta]Y^2 + [\zeta, \zeta]Z^2 + 2[\eta, \zeta]YZ + 2[\zeta, \xi]ZX + 2[\xi, \eta]XY$$

breaks up into the factors

$$[\xi, \xi]\{[\xi, \eta]X + [\eta, \eta]Y + [\zeta, \eta]Z\}$$
$$-\{[\xi, \eta] \pm \sqrt{([\xi, \eta]^2 - [\xi, \xi][\eta, \eta])}\}\{[\xi, \xi]X + [\xi, \eta]Y + [\xi, \zeta]Z\}.$$

Now, since equations (16), (17), (18), (28), (29) are equivalent to only two independent equations,

$$[\xi, \eta]X + [\eta, \eta]Y + [\zeta, \eta]Z \quad \text{and} \quad [\xi, \xi]X + [\xi, \eta]Y + [\xi, \zeta]Z$$

are linear functions of

$$[\xi, \alpha]X + [\eta, \alpha]Y + [\zeta, \alpha]Z \quad \text{and} \quad [\xi, \beta]X + [\eta, \beta]Y + [\zeta, \beta]Z.$$

Hence the equations (183)–(185) may be reduced to others in which the degrees of the lowest terms in X, Y, Z are 3, 1, 1 respectively. Hence there are three intersections.

(B.) If, however, the edge of the binode always touches the binode locus, then by (126) it follows that

$$[\xi, \alpha]X + [\eta, \alpha]Y + [\zeta, \alpha]Z \text{ is a multiple of } [\xi, \beta]X + [\eta, \beta]Y + [\zeta, \beta]Z.$$

In this case

$$[\xi, \eta]X + [\eta, \eta]Y + [\zeta, \eta]Z \quad \text{and} \quad [\xi, \xi]X + [\xi, \eta]Y + [\xi, \zeta]Z$$

are not, as in the last case, linear functions of

$$[\xi, \alpha]X + [\eta, \alpha]Y + [\zeta, \alpha]Z \quad \text{and} \quad [\xi, \beta]X + [\eta, \beta]Y + [\zeta, \beta]Z,$$

for, if so, $[\xi,\xi]\,X+[\xi,\eta]\,Y+[\xi,\zeta]\,Z$ would be a multiple of $[\xi,\eta]\,X+[\eta,\eta]\,Y+[\eta,\zeta]Z$, and the biplanes would coincide, and there would be a uniplanar node.

Consequently, in this case the equations can be reduced as follows :—

The lowest terms in (183) to be of the second degree.

The lowest terms in (184) to be of the first degree.

The lowest terms in (185) by means of (184) to be of the second degree.

Hence the degrees are respectively 2, 1, 2.

Hence there are four intersections.

Art. 29.—*To prove that the Surfaces represented by the three fundamental equations intersect in six points on the Uniplanar Node Locus.*

In this case

$$[\xi,\xi]\,X^2+[\eta,\eta]\,Y^2+[\zeta,\zeta]\,Z^2+2\,[\eta,\zeta]\,YZ+2\,[\zeta,\xi]\,ZX+2\,[\xi,\eta]\,XY$$

is a perfect square, and is proportional to the square of $[\xi,\xi]\,X+[\xi,\eta]\,Y+[\xi,\zeta]\,Z$; and this by means of the ratios (48) is proportional to $[\xi,\alpha]\,X+[\eta,\alpha]\,Y+[\zeta,\alpha]\,Z$ and also to $[\xi,\beta]\,X+[\eta,\beta]\,Y+[\zeta,\beta]\,Z$.

Hence the lowest terms in X, Y, Z may be reduced as follows :—

The lowest terms in (183) to be of the third degree; the lowest terms in (184) to be of the first degree; and the lowest terms in (185) by means of (184) to be of the second degree.

Hence the degrees are 3, 1, 2 respectively.

Hence there are six intersections.

Section VI. (Art. 30).—Exceptional Cases.

Art. 30.

It remains to notice the exceptional cases in which the locus of ultimate intersections is not a surface.

An example is given of each, but the theory is not developed.

The general case which has been considered in this paper is that in which the fundamental equations are satisfied by values of the coordinates which are functions of *both* parameters.

The exceptional cases are :—

(I.) When the fundamental equations are satisfied by values of the coordinates which are functions of *one* parameter.

(II.) When the fundamental equations are satisfied by values of the coordinates which are functions of *neither* parameter, *i.e.*, are independent of the parameters.

(III.) When the fundamental equations cannot be satisfied by any values of the coordinates which make the discriminant or a factor of it vanish, the values of the parameters being finite.

(IV.) When the three fundamental equations, which contain the five quantities x, y, z, a, b are equivalent to two relations only between them.

(V.) When the three fundamental equations, which contain the five quantities x, y, z, a, b are equivalent to one relation only between them.

I. *The Fundamental Equations are satisfied by values of the coordinates which are functions of one parameter only.*

In this case, eliminating the parameter, two relations between the coordinates are obtained. Hence the locus of ultimate intersections is a curve.

Example 18.

Let the surfaces be

$$a^2x^2 - 2abxy + b^2y^2 - 2a(x+1) - 2by + z = 0 \quad . \quad . \quad . \quad (197).$$

(A.) *The Discriminant.*

It is

$$\begin{vmatrix} x^2 & -xy & -x-1 \\ -xy & y^2 & -y \\ -x-1 & -y & z \end{vmatrix}$$
$$= -y^2(2x+1)^2.$$

(B.) The coordinates of each point on the locus of ultimate intersections must satisfy (197) and

$$\left.\begin{aligned} ax^2 - bxy - (x+1) = 0 \\ -axy + by^2 - y = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (198).$$

From (197) and (198)

$$-a(x+1) - by + z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (199).$$

(i.) Now a solution of the second of equations (198) is

$$y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (200).$$

Substituting in the first of equations (198) and in (199)

$$ax^2 - (x+1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (201),$$

$$-a(x+1) + z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (202).$$

From (201) and (202)

$$zx^2 - (x + 1)^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (203).$$

A part of the locus of ultimate intersections is, therefore, given by (200) and (203). In this case x, y, z may be considered to be functions of a only. It will be noticed that if (200) be satisfied, $\Delta = 0$. But $\Delta = 0$ does not suffice to determine this part of the locus of ultimate intersections.

(ii.) Next take the other solution of the second of equations (198), viz. :—

$$- ax + by - 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (204).$$

Combining this with the first of equations (198),

$$2x + 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (205).$$

Hence, by (204),

$$y = (2 - a)/2b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (206).$$

Therefore, by (199),

$$z = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (207).$$

Hence another portion of the locus of ultimate intersections is given by (205) and (207).

In this case the coordinates of any point on the locus of ultimate intersections may be regarded as functions of the single parameter $(2 - a)/2b$.

It will be noticed that if (205) be satisfied, $\Delta = 0$; but $\Delta = 0$ is not sufficient to determine this part of the locus of ultimate intersections.

II. *The Fundamental Equations are satisfied by values of the coordinates which are independent of the parameters.*

In this case all the surfaces of the system pass through a finite number of fixed points, or a fixed curve.

Example 19.

Let the surfaces be

$$\psi(x, y, z) + a\phi(x, y, z) + b\chi(x, y, z) = 0.$$

(A.) To find the locus of ultimate intersections, it is necessary to satisfy at the same time the above, and

$$\phi(x, y, z) = 0,$$

$$\chi(x, y, z) = 0.$$

Hence it is necessary to satisfy

$$\psi(x, y, z) = 0, \quad \phi(x, y, z) = 0, \quad \chi(x, y, z) = 0$$

The locus is, therefore, generally a finite number of points.

The values of x, y, z are independent of the parameters.

(B.) If two of the three expressions ψ, ϕ, χ, say ψ, ϕ, have a common factor θ, then the curve $\theta = 0$, $\chi = 0$ is a part of the locus of ultimate intersections.

(C.) If the equation of the system of surfaces be transformed to plane coordinates, then a point has an equation, and the locus of ultimate intersections would have an equation, which could be determined as a factor of the discriminant.

III. *The Fundamental Equations cannot be satisfied by any values of the coordinates which make the Discriminant vanish, the parameters being finite.*

Example 20.

Let the surfaces be the spheres

$$(z + c)(a^2 + b^2) - (c + d)\{2ax + 2by - x^2 - y^2 - (z - c)(z - d)\} = 0,$$

where c, d are fixed constants; a, b are the parameters.

They all touch the plane $z = d$, and the sphere $x^2 + y^2 + z^2 = c^2$.

(A.) *The Discriminant.*

It is

$$\begin{vmatrix} z + c & 0 & -(c + d)x \\ 0 & z + c & -(c + d)y \\ -(c + d)x & -(c + d)y & (c + d)\{x^2 + y^2 + (z - c)(z - d)\} \end{vmatrix}$$

$$= (c + d)(z - d)(x^2 + y^2 + z^2 - c^2)(z + c).$$

(B.) *The Plane $z - d = 0$ is a part of the Envelope.*

(C.) *The Sphere $x^2 + y^2 + z^2 - c^2 = 0$ is a part of the Envelope.*

(D.) The remaining factor $z + c$ requires explanation. It is on account of this factor that this example is introduced.

If $z + c = 0$, the left-hand side of the equation of the system of surfaces, which is of the second degree in a, b breaks up into two factors, one of the first degree in a, b, the other of degree zero.

But the fundamental equations being equivalent to

$$\begin{aligned} a(z + c) \qquad\qquad & - x(c + d) &= 0, \\ b(z + c) & - y(c + d) &= 0, \\ -a(c + d)x - b(c + d)y + (c + d)&[x^2 + y^2 + (z - c)(z - d)] &= 0, \end{aligned}$$

cannot be simultaneously satisfied by finite values of a, b when $z + c = 0$.

For if $z + c = 0$ and a, b be finite, the equations are equivalent to

$$x = 0, \; y = 0, \quad 2c(c + d)^2 = 0,$$

which equations cannot be satisfied.

Hence the values of a, b are infinite.

IV. *The Fundamental Equations are equivalent to only two relations between the coordinates and parameters.*

In such a case the discriminant must vanish identically.

Example 21.

Let the surfaces be

$$\alpha(x - a)^3 + 3\beta(y - b)^2 = 0,$$

where α, β are fixed constants; a, b the arbitrary parameters.

The other fundamental equations are

$$3\alpha(x - a)^2 = 0,$$
$$6\beta(y - b) = 0.$$

Hence the discriminant vanishes identically.

It may be noticed that in this case each surface of the system has a unodal line. Hence the singularity is of a higher order than when each surface has a single unode.

V. *The Fundamental Equations are equivalent to only one relation between the coordinates and parameters.*

In such a case the discriminant must vanish identically.

Analytically

$$[f(x, y, z, a, b)]^2 = 0$$

is an example.

But the left-hand side is resoluble.

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